Università degli studi di Padova

Dipartimento di Ingegneria dell'Informazione

Tesi di laurea magistrale in Ingegneria dell'Automazione

Local Dynamic Approximate Solution for Stationary Mean Field Games

Relatore: Prof. Maria Elena Valcher - Università degli studi di Padova

Correlatore: Prof. Alessandro Astolfi - Imperial College London

Laureando: Daniele Dan

Anno accademico 2015-2016

Abstract

This thesis studies nonlinear stationary mean field games and provides a way to find an approximate solution of a particular class of problems. In the general case, it is very difficult to find a solution of a mean field game because it requires to solve of a system of partial differential equations. The goal of the work presented in this thesis is to show a procedure to find an approximate local equilibrium for a class of nonlinear mean field games with the cost function of a particular form. The proposed technique is characterized by the fact that it permits to solve two algebraic inequalities instead of a system of PDEs. It is local and formally proved in a neighborhood of the origin. Moreover, we only focus on stationary solutions i.e. functions that describe players behavior due the application of an optimal control after a long time. Finally, a numerical example is introduced and the effectiveness of the proposed technique is shown.

Ringraziamenti

Ringrazio innanzitutto i miei genitori Carla e Flavio, senza di loro tutto questo non sarebbe stato possibile. Grazie.

Vorrei poi ringraziare tutti i miei relatori a partire da Thulasi che mi ha seguito nella preparazione di questa tesi all'Imperial College of London. La ringrazio per la gentilezza e la massima disponibilità che ha sempre avuto nei miei confronti e per tutte le idee e i commenti che sono stati indispensabili nella stesura di questa tesi. Ringrazio inoltre il Professor Astolfi per avermi proposto questo argomento e la Professoressa Valcher per il supporto, la fiducia e i preziosi consigli.

Un enorme ringraziamento va poi a tutti gli amici "storici" Marco, Edoardo, Filippo, Luca, Davide, Claudia, Nicolas, Alessandro, Mattia, Riccardo, per esserci stati sempre, indipendentemente dalla distanza.

Un super grazie ad Angela il sostegno negli ultimi anni e per il supporto nell'ingarbugliato percorso che una tesi all'estero prevede.

Un grazie inoltre a Federica per i lunghi discorsi e le lunghissime camminate nei sei mesi a Londra.

Infine un grazie a tutti coloro che hanno reso questi cinque anni così speciali. Grazie infinite a tutti.

Contents

Notation and Symbols

Note that in all this text the gradient of a function will be considered a row vector. More specifically, let $v(x,t):\ \mathbb{R}^n \times [0,\infty[\longmapsto \mathbb{R}$ be a smooth function where $x = \begin{pmatrix} x_1 & \dots & x_n \end{pmatrix}^T$. Its gradient and consequently its Hessian matrix are always indicated as follows

$\frac{\partial v}{\partial x} = v_x(x,t) = \left(\frac{\partial v}{\partial x_1}, \frac{\partial v}{\partial x_2} \ \cdots \ \frac{\partial v}{\partial x_n} \right) \in \mathbb{R}^n$

$\frac{\partial^2 v}{\partial x^2} = v_{xx}(x,t) = \begin{pmatrix} \frac{\partial^2 v}{\partial x_1^2} & \frac{\partial^2 v}{\partial x_1 x_2} & \cdots & \frac{\partial^2 v}{\partial x_1 x_n} \\ \frac{\partial^2 v}{\partial x_2 x_1} & \frac{\partial^2 v}{\partial x_2^2} & \cdots & \frac{\partial^2 v}{\partial x_2 x_n} \\ \vdots & \vdots & \ddots & \\ \frac{\partial^2 v}{\partial x_n x_1} & \frac{\partial^2 v}{\partial x_n x_2} & & \frac{\partial^2 v}{\partial x_n^2} \end{pmatrix} \in \mathbb{R}^{n \times n}$

This choice is stressed because in literature the gradient is often defined as a column vector. This decision was taken in order to be consistent with the notation used in [15] and [16] on which this work is mainly based.

For the time derivative the classical notation is used, namely

$\frac{dv}{dt} = \dot{v}(x,t) \in \mathbb{R}$

$\frac{\partial v}{\partial t} = v_t(x,t) \in \mathbb{R}$

Let $w(x) : \mathbb{R}^n \longmapsto \mathbb{R}^{1\times n}$ and $W(x) : \mathbb{R}^n \longmapsto \mathbb{R}^{n\times n}$ be two smooth functions. The trace and divergence operators in this text are defined as

$tr\left(W(x)\right) = tr \begin{pmatrix} W_{1,1}(x) & \cdots & W_{1,n}(x) \\ \vdots & \ddots & \vdots \\ W_{n,1}(x) & \cdots & W_{n,n}(x) \end{pmatrix} = \sum_{i=1}^n W_{i,i}(x)$

$div\left(w(x)\right) = div \begin{pmatrix} w_1(x) & \dots & w_n(x) \end{pmatrix} = \sum_{i=1}^n \frac{\partial w_i(x)}{\partial x_i}$

We finally define a new operator called $\bar{\nabla}_x(\cdot)$ that will be used to compute the vectorial derivative of matrix mappings

$$\bar{\nabla}_x\left(W(x)\right) = \begin{pmatrix} \frac{\partial W_{1,1}(x)}{\partial x_1} \\ \vdots \\ \frac{\partial W_{3,1}(x)}{\partial x_1} \end{pmatrix} + \begin{pmatrix} \frac{\partial W_{1,2}(x)}{\partial x_2} \\ \vdots \\ \frac{\partial W_{3,2}(x)}{\partial x_2} \end{pmatrix} + \cdots + \begin{pmatrix} \frac{\partial W_{1,n}(x)}{\partial x_n} \\ \vdots \\ \frac{\partial W_{3,n}(x)}{\partial x_n} \end{pmatrix}$$

Chapter 1

Introduction

This thesis studies nonlinear stationary mean field games and provides a way to find an approximate solution of a particular class of problems. Mean field games are a very recent topic. They are attracting more and more attention, due to the fact that they provide tools to model a lot of important physical, biological and economical phenomena. For example this theory is used to describe particles interactions in relativistic physics as in [49] or in [53] but it is also used to analyzed person-to-person interactions both in a financial framework as in [46] and in a social one as in [5]. In particular, the studies [51] and [52] show that mean field games theory can be very effectively used to model crown dynamics that, especially in last years, is an increasingly important topic.

However, in the general case, it is very difficult to find a solution of a mean field game because it requires to solve of a system of partial differential equations (PDEs). In the literature exact solutions for very specific classes of problems, such as the class of the linear quadratic problems in [4] and [6], are available. Because of this difficulty not only in computing the mean field games solution but also in defining existence and uniqueness conditions for the latter, some authors are focusing on finding procedures to design approximate solutions. However very few works on the latter topic exist. One of them is [51] where a suboptimal result is discussed in a particular practical application. Some approximated solutions for some of the PDEs involved in the solution of a mean field game are also provided in [36] and [30]. Nevertheless there is a need to develop more general methods to find the solution of a larger class of mean field games.

The goal of the work presented in this thesis is to show a procedure to find an approximate local equilibrium for a class of nonlinear mean field games with the cost function of a particular form. The proposed technique is characterized by the fact that it permits to solve two algebraic inequalities instead of system of PDEs. It is local and formally proved in a neighborhood of the origin. Moreover we only focus on stationary solutions i.e. functions that describe players behavior due the application of an optimal control after a long time. However the proposed technique is strongly innovative because no approximate methods for this class of nonlinear mean field game problems are available in literature. The proposed approach is the same used in [15] and [16] but several new concepts are introduced in order to deal with the intrinsically different structure of the equations involved. The theoretical concepts are furthermore implemented in a numerical example in order to illustrate the developed procedure.

In this thesis mean field games are studied as an extension of problems of optimal control and differential games. On one hand, in **Chapter 2,** the optimal control problem is introduced in order to show how a solution can be found by simply solving a PDE. On the other hand, in **Chapter 3,** differential games are briefly described in order to explain what is intended by optimal control in a framework where more than one player is considered. Consequently the concept of Nash equilibrium is defined. In **Chapter 4**, exploiting the previously seen concepts, mean field games are introduced together with the concept of stationary solution. In particular, the two equations, namely the Hamilton-Jacobi-Bellman (HJB) and the Fokker-Planck-Kolmogorov (FPK) equation, which characterize the solution of the mean field game are analyzed. The new concept of algebraic $(P(\cdot), G(\cdot))$ mean field game solution is then defined. It is also formally proved that, using the latter, it is possible to define a solution of an algebraic inequality system that approximates the HJB FPK PDE system. In other words, it permits to find a suboptimal control in a neighborhood of the origin. Finally, a numerical example is introduced and the effectiveness of the proposed technique is shown.

Chapter 2

Infinite Horizon Optimal control

2.1 Introduction

Optimal control theory is a mathematical discipline that deals with the problem of finding how to act on a given *system* so that a certain optimality criterion is achieved. The first works in this area were published about 50 years ago immediately after the World War II. Indeed, with the beginning of the Cold War, the efforts in making use of mathematical theories in defense analyses were increased. Mathematicians in the East and the West almost simultaneously began to develop solution methods for problems which later became known as optimal control problems. An example is the minimum time interception problem for fighter aircraft A detailed description of the origin of this framework is available in [9].

In recent years optimal control theory has had countless applications especially in engineering, physics, mathematics and economy. It studies controllable systems whose dynamics can be modified introducing a controller. The latter may be a human being, an electronic device or anything else and it affects the system by modifying specific quantities called *control variables*. A system is characterized by some variables, called *states*, whose behavior fully describes the system dynamics. Moreover an optimal control problem includes a cost functional that is a function of the states and of the control variables. Generally it must be minimized if it is a cost or maximized if it is a profit. We will see that an optimal control problem can be modeled by a set of differential equations describing the trajectories of the control variables that minimize the cost function.

In this chapter, starting from the mathematical description of a general optimal control problem, the procedure to construct the optimal control law is studied. Furthermore the most important definitions and properties are formally introduced and discussed in detail. Indeed similar theorems and mathematical tools will also be used to deal with differential games and mean-field games. This chapter is focused on the infinite horizon optimal control problem i.e. a problem where states, control laws and cost function are defined in the time interval $[0, \infty[$. This choice is due firstly to the fact that finding a solution of this kind of problem is typically easier because of the absence of a constraint on the terminal state value. Secondly, most of the published papers concerning mean field games, which are the main topic of this work, deal with infinite horizon problems. Besides the solutions of the latter are often used in practice in order to approximate results of problems defined on $[0, T]$ with T very large. For this reason the infinite horizon optimal problem is introduced and the techniques for its resolution are formally explained. Then, in the last section of this chapter, an overview of approximate solutions for the optimal control problem is given. The aim is to explain what is meant by an approximate solution and why it is useful in practice. Moreover, some of the described approaches will be used and deeply analyzed in the context of differential games and mean-field games. In other words the next pages are intended for the reader to get acquainted with the subject and in particular with the techniques that will be used in the following. Moreover they describe the infinite horizon optimal control as a stand-alone argument, while it is often shown as a particular case of the finite horizon optimal control in papers and books.

2.2 Definitions and properties

A general dynamic system can be mathematically described as a Cauchy problem in the state variable $x \in \mathbb{R}^n$ namely as

$$\dot{x} = f(x, t) \qquad x(t_0) = x_0 \tag{2.1}$$

where $t \in \mathbb{R}_+$ is the independent variable, typically the time variable, n is the state space dimension, $x_0 \in \mathbb{R}^n$ is the initial condition at time $t_0 \geq 0$, $f : \mathbb{R}^n \times [0, +\infty[\longmapsto \mathbb{R}$ is a smooth function which describes the behavior of the system. Assuming for simplicity $t_0 = 0$, we define a solution of (2.1) in $\mathbb{R}_+$ as a C^0 curve $\Phi_t(x_0)$

$$x(t) = \Phi_t(x_0) : [0, \infty[\longmapsto \mathbb{R}^n \tag{2.2}$$

such that $x(t)$ satisfies (2.1) for all $t \in \mathbb{R}_+$. The existence and uniqueness of a solution depend the properties of $f(x, t)$. Some necessary conditions are provided by the following theorems.

Theorem 2.1. *[10]****(Global existence and uniqueness)***

Let $f(t, x)$ be

1. piece-wise continuous w.r.t. $t \in \mathbb{R}_+$;

2. globally Lipschitz in $\mathbb{R}^n$;

3. uniformly bounded, in other words there exists $h \in \mathbb{R}_+$ such that $\| f(t, x) \| \leq h$ for all $t \in \mathbb{R}_+$ and $x \subseteq \mathbb{R}^n$.

Then (2.1) admits a unique solution w.r.t. $t \in \mathbb{R}_+$.

Globally Lipschitz conditions are typically hard to satisfy. However, local conditions are enough if the solutions exist in a compact set.

Theorem 2.2. *[10]****(Existence and uniqueness in compact sets)***

Let $f(t, x)$ be

1. piece-wise continuous w.r.t $t \in \mathbb{R}_+$;

2. locally Lipschitz in a set $S \subset \mathbb{R}^n$.

If we know that every solution of (2.1) does not exit $X \subseteq S$ and X is compact, then there is a unique solution w.r.t $t \in \mathbb{R}_+$.

The proofs of Theorems 2.1 and 2.2 are provided in [10]. However, for further information about other types of autonomous dynamical system and stability, see [8].

The aim of control theory is to model and study the presence of an external agent operating on the system typically to reach a specific goal such as to steer the system to a specific location, to stabilize a certain configuration or, as in this case, to maximize a profit or minimize a cost. The agent modifies the dynamics of the system by means of the so called control function $u(t) : \mathbb{R}_+ \longmapsto \mathbb{R}^m$. The controlled dynamic system is mathematically described by a Cauchy problem with the additional presence of the input $u(t) \in U$ where U is the set of all possible input functions

$$\dot{x} = f(x, u, t) \qquad x(t_0) = x_0 \tag{2.3}$$

and hence a solution of (2.3) corresponding to the initial condition x_0 at $t_0 = 0$ and defined for $t \in \mathbb{R}_+$ is

$$x(t) = \Phi_t(x_0, u) \tag{2.4}$$

In this case, as explained in [11] and [12], a necessary condition for the existence and uniqueness of (2.4) is

$$(\mathbf{H1}) \quad \begin{cases} u(t) \text{ is a Lebesgue-measurable function} \\ \\ U \text{ is a compact set} \\ \\ \text{the following holds} \\ \exists K \in \mathbb{R} : \quad |f(x,u,t)| < K(1+|x|) \quad \text{for all (x, u, t)} \in \mathrm{\mathbb{R}^n} \times U \times \mathbb{R}_+ \end{cases}$$

Assumptions **(H1)** furthermore ensure that $x(t)$ is bounded.

In order to determine what is the best control $u^*(t)$, we need to specify a particular cost criterion. Let us define the cost functional

$$J\left(u(t), x_0\right) := \lim_{T \longrightarrow \infty} \int_0^T L\left(x(t), u(t), t\right) dt \tag{2.5}$$

where $x(t)$ solves (2.3) for the control $u(t)$. Here $L(x, u, t) : \mathbb{R}^n \times U \times \mathbb{R}_+ \longmapsto \mathbb{R}$ is given and called running cost. The aim is to find a function $u(t) : \mathbb{R}_+ \longmapsto \mathbb{R}^m$ that permits to minimize $J(u(t), x_0)$ where $x(0) = x_0$ is fixed.

Moreover, it is necessary to ensure the absolute convergence of the integral in (2.5). A necessary condition, which will be referred to as **(H2)** and has to hold for any $u(t) \in U$, is the following

$$(\mathbf{H2}) \begin{cases} \text{Given a control } u(t) \text{ such that } (\mathbf{H1}) \text{ holds and that } x(t) \text{ is} \\ \text{the corresponding solution according to 2.4 and} \\ \text{two positive-valued functions } \mu_1 \text{ and } \mu_2 \text{ on } [0,\infty[\text{ such that} \\ \mu_1(t) \to 0 \text{ , } \mu_2(t) \to 0 \text{ as t} \to \infty\text{, we have} \\ \\ \max_{u \in U} \ |L\left(x(t), u(t), t\right)| \leq \mu_1(t) \text{ for all } t > 0 \\ \\ \int_T^\infty |L\left(x(t), u(t), t\right)| dt \leq \mu_2(t) \ \text{ for all } T > 0 \end{cases}$$

The verification of **(H2)** for all possible controls $u(t) \in U$ may be very laborious or impossible . For this reason, in the scientific literature cost functions of other forms are very often used. The most common formulations are

$$J_{disc,1}\left(u(t), x_0\right) := \lim_{T \longrightarrow \infty} \int_0^T e^{-\alpha t} g\left(x(t), u(t)\right) dt \tag{2.6}$$

$$J_{disc,2}\left(u(t), x_0\right) := \lim_{T \longrightarrow \infty} \frac{1}{T} \int_0^T g\left(x(t), u(t)\right) dt \tag{2.7}$$

where $g\left(x, u\right) : \ \mathbb{R}^n \times \mathbb{R}^m \longmapsto \mathbb{R}$ and $\alpha \in \mathbb{R}$ is a positive constant. (2.6) is referred to as *discounted cost* and (2.7) is referred to as *average cost* and they can model particular economic or physic phenomena. Using (2.6) and (2.7), assumption **(H2)** can be easily verified. Indeed, for instance, having $g\left(x, u\right) : \mathbb{R}^n \times U \longmapsto \mathbb{R}$ bounded is sufficient to prove the absolute convergence of the cost function. The two proposed formulations can however be seen as particular cases of (2.5). This is the reason why only the general case (2.5) will be analyzed in the following. Particular results concerning systems for which these kinds of cost functions are used are provided in [17] and [20].

Finally, a generic infinite horizon optimal control problem is formulated as

$$\begin{cases} \min_{u(t) \in U} \ \lim_{T \longrightarrow \infty} \int_0^T L\left(x(t), u(t), t\right) dt \\ \dot{x} = f(x, u, t,) \qquad x(t_0) = x_0 \end{cases} \tag{2.8}$$

Note that the optimization has to be performed in the infinite dimensional space U. So, it is very difficult to find a minimum by directly solving (2.8). For this reason two important tools, i.e. the Pontryagin Minimum Principle and the Hamilton Jacobi Bellman PDE, will be introduced in the next section. These provide necessary and sufficient conditions for $u(t)$ to be optimal.

2.3 The Pontryagin Minimum Principle

The most important and powerful tool to search for an explicit solution of an optimal control problem is the well known Pontryagin Minimum Principle. It was introduced by Pontryagin in 1962. The classical formulation is provided in the original paper [13] or in [14] where the typical control theory notation is used. In its original form the principle aimed at solving a maximization problem. However here it is formulated for the previously defined minimization problem (2.8). It can be thought of as the equivalent of the Lagrange Multiplier Method for optimization in an infinite dimensional space. Indeed (2.8) can be seen as a problem of constrained optimization. The functional we have to minimize is (2.5) and the dynamic equation (2.1) plays the role of the constraint.

First of all we need to give the definition of Hamiltonian function that is the key concept the minimum principle is based on.

Definition 2.3. [14]**(Hamiltonian Function)** Given the optimal control problem (2.8), $p_0 \in \mathbb{R}$ and $p(t) : [0, \infty[\to \mathbb{R}^{1\times n}$, define *Hamiltonian function* $\mathcal{H} : \mathbb{R}^n \times \mathbb{R}^m \times [0, \infty[\times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ the following

$$\mathcal{H}(x, u, t, p, p_0) = p(t)\, f(x, u, t) + p_0 L(x, u, t)$$

Accordingly, the minimized Hamiltonian function can be defined as follows.

Definition 2.4. [14]**(Minimized Hamiltonian Function)** Given the optimal control problem (2.8), $p_0 \in \mathbb{R}$ and $p(t) : [0, \infty[\to \mathbb{R}^n$, the *minimized Hamiltonian function* $H : \mathbb{R}^n \times [0, \infty[\times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is defined as follows

$$H(x, t, p, p_0) = \inf_{u \in U} \mathcal{H}(x, u, t, p, p_0) \tag{2.9}$$

In the literature and in this work the classical notation $H(x, t, p)$, corresponding to $H(x, t, p) = H(x, t, p, 1)$, will be used. Moreover, with some misuse of terminology, $H(x,t,p)$ will simply be referred to as the Hamiltonian function.

Theorem 2.5. ***[19](Pontryagin Minimum Principle)*** *Call $u^*(t)$ the solution of the problem (2.8) and $x^*(t)$ the corresponding trajectory.*

*If **(H1)** and **(H2)** hold for $(u^*(t), x^*(t))$, then there exists a couple $(p(t),\ p_0)$, called "couple of adjoint variables associated with $(u^*(t), x^*(t))$", where $p_0 \in \mathbb{R}$ and $p:\ [0,\infty[\to \mathbb{R}^n$ is a continuous piecewise differentiable function such that*

1. $\dot{p}(t) = -p(t)\left[\frac{\partial f(x^*(t),u^*(t),t)}{\partial x}\right] - p_0\left[\frac{\partial L(x^*(t),u^*(t),t)}{\partial x}\right]$
2. $H(x^*(t), t, p, p_0) = \mathcal{H}(x^*(t), u^*(t), t, p, p_0) \quad \forall t \geq 0$
3. $\| p(0) \| + p_0 > 0$

A complete proof is provided for instance in [22], [19] and [21] where the different formulation of the cost functional (2.6) is also considered. For the sake of completeness, the Pontryagin Maximum Principle is stated below.

Theorem 2.6. ***[19](Pontryagin Maximum Principle)***

Given the following problem

$$\begin{cases} \max_{u(t)\in U}\ \lim_{T\longrightarrow\infty} \int_0^T L(x(t), u(t), t)\, dt \\ \dot{x} = f(x, u, t,) \qquad x(t_0) = x_0 \end{cases}$$

Call $u^*(t)$ its solution and $x^*(t)$ the corresponding state trajectory.

*If **(H1)** and **(H2)** hold for $(u^*(t), x^*(t))$, then there exists a couple $(p(t),\ p_0)$, called "couple of adjoint variables associated with $(u^*(t), x^*(t))$", where $p_0 \in \mathbb{R}$ and $p(t):\ [0,\infty[\to \mathbb{R}^n$ is a continuous piecewise differentiable function such that*

1. $\dot{p}(t) = -p(t)\left[\frac{\partial f(x^*(t),u^*(t),t)}{\partial x}\right] - p_0\left[\frac{\partial L(x^*(t),u^*(t),t)}{\partial x}\right]$
2. $\sup_{u\in U} \mathcal{H}(x, u, t, p, p_0) = \mathcal{H}(x^*(t), u^*(t), t, p, p_0) \quad \forall t \geq 0$
3. $\| p(0) \| + p_0 > 0$

Consider now the following assumption that will be referred to as **(H3)**

$$
(\mathbf{H3}) \begin{cases} \text{For each } x \in \mathbb{R}^n \text{ the function } u \to f(x,u,t) \text{ is affine i.e.} \\ \\ f(x,u,t) = f_0(x) + f_1(x)u(t) \text{ for all } x \in \mathbb{R}^n \text{ and all } u \in U \\ \\ \text{where } f_i : \mathbb{R}^n \to \mathbb{R}^n,\, i = 0,1 \text{ are continuously differentiable} \end{cases}
$$

According to [17], if **(H3)** holds, we can fix $p_0 = 1$ in Theorem (2.5) and obtain the following formulation.

Theorem 2.7. ***[17](Pontryagin Minimum Principle - Normal Form)*** *Call $u^*(t)$ the optimal solution of the problem (2.8) and $x^*(t)$ the corresponding trajectory.*

If ***(H1)****,* ***(H2)*** *and* ***(H3)*** *hold for $(u^*(t), x^*(t))$, then there exists a continuous piecewise differentiable function $p : [0, \infty[\longmapsto \mathbb{R}^n$ such that*

1. $\dot{p}(t) = -p(t)\left[\frac{\partial f(x^*(t),u^*(t),t)}{\partial x}\right] - \left[\frac{\partial L(x^*(t),u^*(t),t)}{\partial x}\right]$
2. $H\left(x^*(t), t, p, 1\right) = \mathcal{H}\left(x^*(t), u^*(t), t, p, 1\right) \quad \forall t \geq 0$

A counter-example, that shows that we can not normalize the Hamiltonian by assuming $p_0 = 1$ unless **(H3)** hold, is proposed in [17].

Theorem 2.7 provides just some necessary conditions for $u^*(t)$ to satisfy. This means that it suggests a way to find all the optimal control candidates. In detail, it consists of the following steps

1. Find one of the possible candidates $u_C(t,x,p)$ such that

   $$u_C(t,x,p) = \arg\min_{u \in U} \mathcal{H}\left(x,u,t,p,1\right)$$

   where we remark that $u_C(t,x,p)$ depends also on p and it may be discontinuous.

2. Solve the problem

   $$\begin{cases} \dot{x} = f\left(x(t), u_C(t), t\right) & x(t_0) = x_0 \\ \dot{p}(t) = -p(t)\left[\frac{\partial f(x(t),u_C(t),t)}{\partial x}\right] - \left[\frac{\partial L(x(t),u_C(t),t)}{\partial x}\right] \end{cases} \tag{2.10}$$

that can be thought of as an extended system that has $z(t) = (x(t), p(t))^T$ as its state, $z(t) = (x_0, 0)^T$ as its initial condition and
$u_C(t, x(t), p(t)) = u_C(t, z(t))$ as its input. If a solution $(x_C(t), p_C(t))$ of (2.10) exists, it satisfies all the conditions of the Pontryagin minimum principle. For this reason $u\left(t, x_C(t), p_C(t)\right)$ is a candidate optimal control.

Theorem (2.7) produces necessary but not sufficient conditions for optimality. This means that it permits to find a set S of possible optimum candidates $u\left(t, x_C(t), p_C(t)\right)$ that have to be tested one by one. Moreover this procedure may be impossible to implement because S may have infinitely many elements. Under more specific hypotheses on the cost function (2.5), the PMP becomes a necessary and sufficient condition as it will be seen in the following. We define the Assumption **(H4)** as follows

(**H4**)For each $x \in \mathbb{R}^n$ the function $u \to L\left(x(t), u(t), t\right)$ is convex

Theorem 2.8. *[22] If $u^*(t) \in U$ is an input for the system (2.8) and $x^*(t)$ is the corresponding state trajectory according to (2.3) and if the following conditions hold*

*1. **(H1)**, **(H2)**, **(H3)** and **(H4)** hold for $(u^*(t), x^*(t))$*

2.$u^*(t)$ is such that

$$H\left(x^*(t), t, p, 1\right) = \mathcal{H}\left(x^*(t), u^*(t), t, p, 1\right) \quad \forall t \geq 0$$

3.the solution $p(t)$ of

$$\dot{p}(t) = -p(t)\left[\frac{\partial f\left(x^*(t), u^*(t), t\right)}{\partial x}\right] - \left[\frac{\partial L\left(x^*(t), u^*(t), t\right)}{\partial x}\right]$$

is such that

$$\lim_{t\to\infty} \parallel p(t) \parallel = 0$$

Then $u^*(t)$ is the optimal control for the problem (2.8).

A proof is provided in [22]. Analyzing the definition of Hamiltonian function, the infinite horizon optimal control problem (2.8) can be rewritten as a boundary value problem as follows

$$\begin{cases} \dot{x} = \frac{\partial H}{\partial p}(x, t, p, 1) & x(t_0) = x_0 \\ \dot{p} = -\frac{\partial H}{\partial x}(x.t, p, 1) & \lim_{t\to\infty} \parallel p(t) \parallel = 0 \end{cases} \tag{2.11}$$

In other words, if assumptions **(H1)**, **(H2), (H3)** and **(H4)** hold, solving (2.8) is equivalent to solving the system of PDEs (2.11) in the unknown variables $p(t)$ and $x(t)$. In this case the optimal control exists, is unique and is given by

$$u^*(t) = \arg\min_{u\in U} \mathcal{H}(x(t), u, t, p(t), 1)$$

Besides note that in this problem, differently from the finite horizon case, we do not have a constraint on the final value of $p(t)$ but we need to check the so called *transversality condition*

$$\lim_{t\to\infty} \parallel p(t) \parallel = 0$$

To sum up, in the particular case of $\lim_{t\to\infty} \parallel p(t) \parallel > 0$, we can not say if the solution of (2.11) is optimal or not without further assumptions. This condition can be eliminated or made easily verifiable using another kind of cost functional such as (2.6), see, for example, [17].

As we have seen, this procedure is often difficult to apply because the hypotheses of the Pontryagin principle are difficult to verify. This is the reason why it is not typically used in the infinite horizon problems.

2.4 Hamilton Jacobi Bellman PDE

Theorem 2.8 provides a sufficient condition for optimality if **(H1)**, **(H2), (H3)** and **(H4)** hold. Such assumptions, in particular **(H3)** and **(H4),** are far too restrictive. Besides, in several applications, they are not satisfied. In this section we introduce the Hamilton Jacobi Bellman PDE, which constitutes a sufficient condition for optimality, regardless of the structure of the dynamics $\dot{x} = f(x, u, t)$ and of the cost function $J(u(t), x_0)$. We briefly described the optimal control problem (2.8) with fixed initial condition $x(t_0) = x_0$ so far. Now we introduce a value function specifying the best possible value of the cost function starting from each state.

Definition 2.9. [33]**(Value Function)** If **(H2)** holds for each $u(t) \in U$, we define *value function* the function $V:\ \mathbb{R}^n \times [0,\infty[\longmapsto \mathbb{R}$

$$V(x,t) = \inf_{u(t)\in U} J\left(u(t),x,t\right)$$

where U is the set of all possible inputs, $J\left(u(t),x_0,t_0\right)$ is the cost function defined in (2.5) with $x_0 = x \in \mathbb{R}^n$ and $x(t)$ is the solution of the Chauchy problem (2.3) for $x(t_0) = x$ and $t_0 = t \geq 0$.

Note that **(H2)** ensures that $V(x_0,t_0)$ is finite. However, without any further assumption, there may be more than one $u(t) \in U$ such that $J\left[u(t),x,t\right] = V(x,t)$ with x and t fixed.

We now introduce the dynamic programming principle. It describes an important property of the value function, based on which the Hamilton Jacobi Bellman PDE will be derived in the following.

Theorem 2.10. ***[33](Dynamic Programming Principle)***

For each $x_0 \in \mathbb{R}^n$, $t_0 \geq 0$ and $h \geq 0$, if ***(H1)*** *holds and* ***(H2)*** *holds for each $u(t) \in U$, the value function satisfies*

$$V(x_0,t_0) = \inf_{u(t)\in U}\left[\int_t^{t+h} L\left(x(s),u(s),s\right)ds + V(x(t_0+h),t+h)\right] \quad (2.12)$$

where $x(s) = x(s;t_0,x_0,u(t))$ and $x(t_0+h) = x(t_0+h;t_0,x_0,u(t))$.

The proof, which can be found in detail in [33], is given in the Appendix B. It is important in order to understand how the previously defined Hamiltonian function is linked to the optimal control function $u(t)$.

The dynamic programming principle can be interpreted as follows. Consider an optimal control $u^*(t)$ and the correspondent trajectory $x^*(t)$ that solves (2.3) with $t_0 = \hat{t}$ and $x_0 = \hat{x}$. Then, the minimum cost $V\left(\hat{t},\hat{x}\right)$ is equal to the running cost from $\hat{t}$ to a generic $\bar{t} > \hat{t}$ following the optimal trajectory $x^*(t)$ plus the minimum cost $V\left(\bar{t},x^*(\bar{t})\right)$, namely

$$V\left(\hat{t},\hat{x}\right) = \int_{\hat{t}}^{\bar{t}} L\left(x^*(s),u^*(s),s\right)ds + V\left(\bar{t},x^*(\bar{t})\right)$$

Exploiting this idea it is possible to split the whole optimization problem into infinitesimal problems to obtain the Hamilton Jacobi Bellman PDE.

Theorem 2.11. ***[33](Hamilton Jacobi Bellman PDE - necessary condition)***

Consider the optimal control problem (2.8) and the Hamiltonian function (2.9). If ***(H1)*** *holds and* ***(H2)*** *holds for each $u(t) \in U$ and $V(t,x) : [0,\infty[\times\mathbb{R}^n \longrightarrow \mathbb{R}$ is the value function of (2.8) , then on the region Ω where V is differentiable*

$$V_t + H(t, x, V_x) = 0 \tag{2.13}$$

The complete proof is provided in Appendix B.

Theorem 2.12. ***[33](Hamilton Jacobi Bellman PDE - sufficient condition)***

Consider the optimal control problem (2.8) and the Hamiltonian function (2.9). If ***(H1)*** *holds and* ***(H2)*** *holds for each $u(t) \in U$ and $\tilde{V}(t,x) : [0,\infty[\times\mathbb{R}^n \longrightarrow \mathbb{R}$ is a differentiable function such that*

$$\tilde{V}_t + H\left(t, x, \tilde{V}_x\right) = 0 \tag{2.14}$$

on a region Ω, then $\tilde{V}(t,x)$ is the value function of (2.8) in Ω.

The proof is similar to the one of Theorem 2.11 and can be found in [33]. It also provides a formula to compute an optimal control function. In particular it shows that, if $V(t,x) : [0,\infty[\times\mathbb{R}^n \longrightarrow \mathbb{R}$ is the value function, then $u^*(t,x) : [0,\infty[\times\mathbb{R}^n \longmapsto \mathbb{R}^m$ is an optimal control if

$$u^*(t,x) = \arg\min_{u(t)\in U} \left(V_x(t,x) f(x,u,t) + L(x,u,t)\right) \tag{2.15}$$

However $u^*(t,x)$ may not exist and may not be unique.

The differences between HJB PDE theorem and Pontryagin minimum principle are summarized in the following

- Using (2.15), we obtain a *feedback* optimal control $u^*(t,x)$ because of the dependence on the state x. In this case, if the initial condition x_0 changes , we do not have to recompute the solution, but we simply need to evaluate $u^*(t,x)$ at a different point. On the contrary, the optimal control computed by the Pontryagin Minimum Principle, see Theorem 2.8, depends only on t.

- If we use (2.15) in order to computer the optimal control, we first need to calculate the value function gradient $V_x(t,x)$ from the Hamilton Jacobi Bellman PDE (2.14). On the other hand, using the Pontryagin Minimum Principle, we only need to solve a set of ordinary differential equations, that need to be satisfied only on a specific trajectory.
- The Pontryagin Minimum Principle produces sufficient conditions only under restrictive convexity conditions. Indeed it needs **(H1)**, **(H2)** ,**(H3)** and **(H4)** to hold. The scope of the HJB PDE is more general and therefore it only requires that **(H1)** and **(H2)** hold and that the candidate value function $\tilde{V}(t,x)$ is differentiable on a region Ω.

Moreover, some studies prove that Theorem 2.12 can be generalized considering not differentiable candidate value functions $\tilde{V}(t,x)$. In that case (2.14) is substituted by a set of inequalities and the solution is referred to as *viscosity solution*, see [32] and [34].

2.5 Approximate Solutions

In the previous section we have showed that, if particular assumptions hold, solving an infinite horizon optimal control problem is equivalent to solving the HJB PDE (2.14). An explicit feedback solution is provided only in specific situations such as in linear problems [35] or when the Hamiltonian function has a particular form [37]. In the general case, it may be hard or impossible to determine an explicit solution. Hence, the problem of finding approximate solutions to the PDEs arising from nonlinear control problems becomes very important. The purpose of this section is to give an idea of the approximate solutions proposed in the literature.

In recent years, several ways to estimate the solution of the Hamilton–Jacobi–Bellman PDE, in a neighborhood of an equilibrium point, have been proposed. For instance in [30] a local solution is calculated and conditions for its existence, for a parametrized family of infinite horizon optimal control problems, are given. On the other hand, other techniques are focused on the linearization of the system. For example in [31] the proposed local solution hinges upon the repeated computation of the steady-state solution of the Riccati equation. The key idea of the state-dependent Riccati equation is to calculate the Riccati equation pointwise, using a state-dependent linear representation (see [39]). Another approach consists in the problem linearization, the computation of the solution and of the explicit error estimates which have to be minimized, see [36].

Finally, the procedure proposed in [15] permits to design a dynamic, time-varying, suboptimal solution of the HJB equation. In particular the infinite horizon optimal control problem with a stability constraint is considered. As we will show, the same steps shown in [15] will be adopted in the derivation of the approximate mean field games solution that is the main topic of this work.

Chapter 3

Differential Games

3.1 Introduction

Differential games theory is a branch of game theory that studies problems related to the modeling and analysis of conflict in the context of one or more dynamical systems. It can be thought of an extension of optimal control theory where more than one agent and consequently more than one cost function are considered. Here again, early analyses on this matter were related to military interests. The study of differential games was initiated by R. Isaacs and S. Pontryagin in the middle 1950s and it was motivated by pursuit and evasion problems. In recent years this theory has been of interest to a big range of applications. It is still used for military purposes. For instance in [23] the problem is linked to the correct choice of the trajectories of a storm of aircrafts in an air operation. However it is also used in economic and management, for example in [24] where a global taxation scheme is modeled with the goal of limiting environmental pollution of the various states. Another field of study where this theory is deeply exploited is biology. For instance in [25] a more formal formulation of the theory of evolution is given. In particular every living being is modeled as a player of a game where the purpose is the survival of the species. Finally, differential games theory is also applicable to problems involving multi-agent systems, which are widely studied in recent years, see [26].

In this chapter we want to give an overview of differential game theory. Moreover we intend to show the different formulations of the problem and to provide the mathematical tools required for its solution. In particular we intend to focus on nonzero-sum games e.g. games in which the gain of a player is not necessary linked to the loss of another one. This choice is made because mean field games, the main topic of this work, can precisely be thought of a class of nonzero sum differential games. Infinite time horizon will be also considered for the same reasons given in the previous chapter. Finally, in the section 3.4 approximate solutions are introduced. Their meaning is explained and the technique used for their computation is analyzed.

3.2 Game Theory

In order to give to the reader a better understanding of the new issues emerging when we study a problem of differential games, it is helpful to analyze first some simple 2-player games. Indeed the issues emerging in this reduced framework are the same ones that characterize N-player differential games.

Definition 3.1. [28](2-Player Classical Game) A 2-player game involves 2 rational agents called players p_1 and p_2 and is defined by the 4-tuple $\{S_1, S_2, f_1(\cdot,\cdot), f_2(\cdot,\cdot)\}$ where

- for $i = 1, 2$, $S_i \subseteq \mathbb{R}$ is a set from which the player p_i can choose an element u_i called *strategy;*
- for $i = 1, 2$, $f_i(u_1, u_2) : S_1 \times S_2 \longmapsto \mathbb{R}$ is the function that the player p_i has to minimize with his choice of u_i.

If $f_1(u_1, u_2) + f_2(u_1, u_2) = 0$ the game is called zero-sum game. In this case the gain of a player corresponds to the loss of the other one. In other words their goals are completely opposite because $f_1(u_1, u_2) = -f_2(u_1, u_2)$. However, in some games one can encounter "win-win situations" which result in a positive outcome for both parties. Such frameworks, and many others, are not captured by zero-sum games. For this reason we will study the more general case of nonzero sum games. In this situation the players' goals can be unrelated. In that way they do not have to be in competition with each other and they may play cooperatively, if it is beneficial.

Note that in this framework it is in general very difficult to find strategies u_1 and u_2 that permits to minimize both $f_1(\cdot,\cdot)$ and $f_2(\cdot,\cdot)$ simultaneously. For this reason we have to introduce the concept of *equilibrium.*

Definition 3.2. [28](Nash Equilibrium) A pair $(\bar{u}_1, \bar{u}_2) \in S_1 \times S_2$ is called a Nash equilibrium of the problem posed in Definition 3.1 if

$$\begin{cases} f_1(\bar{u}_1, \bar{u}_2) \leq f_1(u_1, \bar{u}_2) & \forall u_1 \in S_1 \\ f_2(\bar{u}_1, \bar{u}_2) \leq f_2(\bar{u}_1, u_2) & \forall u_2 \in S_2 \end{cases}$$

The Nash equilibrium concept was first studied by A.Cournot in [40] in the oligopoly theory. Then it was investigated by J. Nash who proved the existence of such equilibrium in the particular N-person game proposed in [41]. The main feature of this solution concept is that neither of the players can decrease his cost by changing unilaterally his own strategy, as long as the other player sticks to the equilibrium solution.

Roughly speaking, in a Nash equilibrium situation, a player is assumed to know the equilibrium strategy of the other one and it has nothing to gain by changing his own strategy. Obviously this concept, as we will see, can be extended also to N-player games and N-player differential games. Note that, in addition to Nash equilibrium, other kinds of equilibrium conditions are defined in literature, see, for example, [18] and [28]. Some of them are, for instance,

- *minimax* that is the pair $(\bar{u}_1, \bar{u}_2)$ where $\bar{u}_1$ is the best strategy that player p_1 can choose in order to minimize his cost assuming $\bar{u}_2$ is the worst possible. The same holds with symmetry for $\bar{u}_2$.

- *Pareto optimality* that is the pair $(\bar{u}_1, \bar{u}_2)$ such that there exists no other pair $(u_1, u_2) \in S_1 \times S_2$ such that

$$f_1(u_1, u_2) \leq f_1(\bar{u}_1, \bar{u}_2) \quad i = 1, 2$$

  and at least one of the inequalities is strict.

Nevertheless the purpose of this chapter is to provide the key concepts useful for the comprehension of mean field games. In the mean field games framework a solution is considered optimal when Nash equilibrium conditions hold. For this reason we do not go further talking about other equilibrium conditions and we focus only on Nash equilibrium. We in particular stress that Nash equilibrium is not the set of the strategies that minimize each player cost function. Moreover it does not necessarily exist or, on the contrary, Nash equilibrium might exist but not be unique. In that case, as it is shown in some examples in [28], different Nash equilibria can yield different costs for each player.

3.3 Definitions and properties

The issues that we have showed, for the sake of simplicity, for 2-player game characterize also the N-player game and the N-player differential game. The latter, in particular, differs from a classical game like the one described in Definition 3.1 because the strategies u_i are substituted by smooth functions $u(t) : [0, \infty[\longmapsto \mathbb{R}^b$. Roughly speaking, each player can change his choice at every instant t.

Definition 3.3. [18](N-player Differential game) A N-player differential game consists on the following elements

- a finite set of players P where $|P| = N$ and hence P can be described as $P = \{1, 2, 3 \ \dots \ N\}$;
- the set $\mathcal{X}$ of the bounded smooth functions $x(t) : [0, \infty[\longmapsto \mathbb{R}^d$ where d is the state dimension;
- the set U of the bounded smooth functions $u(t) : [0, \infty[\longmapsto \mathbb{R}^b$ where b is the control input dimension;
- two injective functions $\Gamma_1 : P \longmapsto \mathcal{X}$ and $\Gamma_2 : P \longmapsto \mathcal{U}$ such that each player $p \in P$ is associated with both a state $x^p(t)$ and an input $u^p(t)$;
- N known dynamic equations of the form

  $$\dot{x}^i(t) = f^i(x^i(t), u^1(t), \dots \ u^N(t), t) \quad \forall i = 1, 2, \ \dots \ N$$

  that describe how the state of each player i changes;
- N control functions $u^i(t)$;
- N known cost functions $J^i \left[x^1, x^2, \ \dots \ x^N, u^1, u^2, \ \dots \ u^N\right]$ of the form

  $$J^i = \lim_{T \longrightarrow \infty} \int_0^T L^i \left(x^1, x^2, \ \dots \ x^N, u^1, u^2, \ \dots \ u^N\right) dt \tag{3.1}$$

  where $L : \mathbb{R}^{dN} \times \mathbb{R}^{bN} \times [0, \infty[\longmapsto \mathbb{R}$ smooth and $1 \leq i \leq N$;

- N known initial conditions $\left(x_0^1, x_0^2, \dots x_0^N\right)$ with $x_0^i \in \mathbb{R}^d$ and N known initial instants $\left(t_0^1, t_0^2, \dots t_0^N\right)$ with $t_0^i \in \mathbb{R} \quad \forall i = 1, 2, \dots N$ such that

$$x_0^i = x^i(t_0^i) \quad \forall i = 1, 2, \dots N$$

- for each player $1 \leq i \leq N$ the problem

$$\begin{cases} \min_{u^i(t) \in U} \lim_{T \longrightarrow \infty} \int_0^T L^i\left(x^1, x^2, \dots x^i, \dots x^N, u^1, u^2, \dots u^i, \dots u^N\right) dt \\ \dot{x}^i(t) = f^i(x^i(t), u^1(t), \dots u^N(t), t) \qquad x^i(t_0^i) = x_0^i \end{cases} \tag{3.2}$$

  to solve by choosing an appropriate $u^i(t)$ or $u^i(x^i, t)$ in case of feedback strategies.

Note that a differential game is typically described only by the expression (3.2) in order to simplify the notation.

Moreover, even in this case, the cost function can be substituted with the average cost (2.7) or the discounted cost (2.6) depending on the situation that has to be modeled.

Watching the definition and in particular system (3.2), the link between differential games problems and optimal control problems is clear. Indeed the latter, described by (2.8), can be thought of a 1-player differential game. In this context the set of the control inputs $\left(u^1(t), \dots u^N(t)\right)$ is still called *strategy*. Moreover, like in classical games, and differently from optimal control problems, we have to define what we mean by optimal solution. As we notice from the dependencies of the cost function $J^i\left(x^1, x^2, \dots x^N, u^1, u^2, \dots u^N\right)$, the gain of the i-th player depends, in general, on the state of the other players $\left(x^1, x^2, \dots, x^{i-1}, x^{i+1}, \dots x^N\right)$ and on their control inputs $\left(u^1, u^2, \dots, u^{i-1}, u^{i+1}, \dots u^N\right)$ too. Therefore, as we saw in the previous section, we need to define the Nash equilibrium concept.

In case of optimal control we did not dwell on the differences between open-loop and closed-loop solutions. Nevertheless, it is necessary to do it for differential games. Indeed, exactly as in the optimal control case, on one hand Pontryagin minimum principle can be used to find necessary conditions for the optimality of a vector $\left(u^1(t), u^2(t), \dots u^N(t)\right)$ for the open-loop system. On the other hand HJB PDE can be exploited to find sufficient conditions for the optimality of a vector $\left(u^1(x), u^2(x), \ldots u^N(x)\right)$ for the closed-loop system. Finally note that the terms "optimum", "solution" and "equilibrium" will be used interchangeably with the same meaning.

In order to make the notation easier, a new column vector

$$X(t) = \left(\left(x^1(t)\right)^T, \ldots \left(x^i(t)\right)^T, \ldots \left(x^N(t)\right)^T\right)^T \in \mathbb{R}^{dN}$$

is defined. It consists of the piled states of all players and it gives a complete information about the differential game state in a particular instant. Consequently the initial condition set $\left(x_0^1, x_0^2, \ldots x_0^N\right)$ is also called

$$X_0 = \left(\left(x_0^1\right)^T, \ldots \left(x_0^i\right)^T, \ldots \left(x_0^N\right)^T\right)^T \in \mathbb{R}^{dN}$$

and $T_0 = \left(t_0^1 \ldots t_0^N\right) \in \mathbb{R}^N$ are the instants which we know the initial condition x_0^i for each player i at.

We also remark that the state is often shared i.e. $x^i(t) = x^j(t)$ for each $i, j = 1 \ldots N$. Refer to [7] for a complete background on differential games.

3.3.1 Open-loop Nash Equilibria and Admissible Strategies

Definition 3.4. [18](Open-Loop Nash Equilibrium) A vector of control functions $\left(\bar{u}^1(t), \bar{u}^2(t), \ldots \bar{u}^N(t)\right)$ is a Nash equilibrium for a differential game described in 3.3 if the following condition holds for each player i.e. for every $1 \leq i \leq N$

- The control $\bar{u}^i(t)$ is a solution to the optimal control problem for the i-th player with fixed $\left(u^1, \ldots u^{i-1}, u^{i+1}, \ldots u^N\right) = \left(\bar{u}^1, \ldots \bar{u}^{i-1}, \bar{u}^{i+1}, \ldots \bar{u}^N\right)$, namely

$$\begin{cases} \min_{u^i \in U} \int_0^\infty L^i\left(x^1, \ldots x^N, \bar{u}^1, \ldots u^i, \bar{u}^{i+1}, \ldots \bar{u}^N\right) dt \\ \dot{x}^i(t) = f^i(x^i(t), \bar{u}^1(t), \ldots u^i(t), \ldots, \bar{u}^N(t), t) \qquad x^i(t_0^i) = x_0^i \end{cases}$$

This definition of Nash equilibrium itself poses a N-player differential game as N optimal control problems that are coupled. Therefore the techniques we have shown for optimal control problems can be used to formulate necessary conditions for Nash optimality. In particular, in the open-loop case, Pontryagin Minimum Principle can be applied. If **(H1)**, **(H2)** and **(H3)** hold for each $1 \leq i \leq N$, then a optimum candidate $\left(\widetilde{u}^1(t), \widetilde{u}^2(t), \ \dots \ \widetilde{u}^N(t)\right)$ can be found using this technique:

1. Compute the vector $\left(\widetilde{u}^1(t, x^1, \dots x^N, p^1), \dots \widetilde{u}^N(t, x^1, \dots x^N, p^N)\right)$ such as

$$\widetilde{u}^i\left(t, x^1 \dots x^N, p^i\right) = \\ \arg\inf_{u^i \in U} \left\{p^i f^i(x^i, u^i, t) + L^i\left(x^1, \dots x^N, \widetilde{u}^1, \dots, u^i, \dots \widetilde{u}^N\right)\right\}$$

   for each $1 \leq i \leq N$. Note that an infimum may not exist or, if it exists, it may not be unique.

2. According to the Theorem 2.7, substitute the vector $\left(\widetilde{u}^1(t, x^1, \dots x^N, p^1), \dots \widetilde{u}^N(t, x^1, \dots x^N, p^N)\right)$ in the following system of $3N$ equations

$$\begin{cases} \dot{p}^i(t) = -p^i(t)\left[\frac{\partial f^i\left(x^i(t), \widetilde{u}^1, \dots \ \widetilde{u}^i(t, x^1, \dots x^N p^i), \dots \ \widetilde{u}^N, t\right)}{\partial x^i}\right] - \left[\frac{\partial L^i\left(x^1, \dots x^N, \widetilde{u}^1, \dots \widetilde{u}^N\right)}{\partial x^i}\right] \\ \dot{x}^i(t) = f^i\left(x^i(t), \widetilde{u}^1, \dots \ \widetilde{u}^i(t, x^1, \dots x^N p^i), \dots \ \widetilde{u}^N, t\right) \\ x^i(t_0^i) = x_0^i \end{cases} \tag{3.3}$$

   for each $1 \leq i \leq N$.

3. If the system (3.3) has a solution $\left(\hat{x}^1(t), \dots \hat{x}^N(t), \hat{p}^1(t), \dots \hat{p}^N(t)\right)$, then the candidate optimal solution $\left(\widetilde{u}^1(t), \widetilde{u}^2(t), \ \dots \ \widetilde{u}^N(t)\right)$ is given by

$$\widetilde{u}^i(t) = \widetilde{u}^i\left(t, \hat{x}^1(t), \dots \hat{x}^N(t), \hat{p}^i(t)\right)$$

   for each $1 \leq i \leq N$.

As we have shown, Pontryagin Minimum Principle provides just necessary conditions for a solution of the differential game. For this reason it may be impossible to use the previous procedure because, for instance, system (3.3) may have no solution. Furthermore, even if a candidate $\widetilde{u}^1(t) = \widetilde{u}^i\left(t, x^1(t), \dots x^N(t), p^i(t)\right)$ can be found, it may not be a Nash equilibrium.

3.3.2 Closed-loop Nash Equilibria and Admissible Strategies

Because of this lack of sufficient conditions for the optimal solutions, differential games problems are usually solved, when it is possible, using feedback control that is by resorting to a control function with an explicit dependence on the system state, namely $u^i(t, x^1, \dots x^N) : [0, \infty[\times \mathbb{R}^{dN} \longrightarrow \mathbb{R}^b$. Closed-loop control can be implemented only if each player i can observe all players states $X(t) = \left((x^1(t))^T, \dots (x^i(t))^T, \dots \left(x^N(t)\right)^T \right)^T$. For this kind of problems, as we have shown in Section 4.3, it is possible to apply the Theorem 2.12 in order to obtain sufficient conditions for optimality.

The definition of the closed loop Nash equilibrium is given in the following.

Definition 3.5. [18](Closed-Loop Nash Equilibrium) A vector of control functions $\left(\bar{u}^1(t,X), \bar{u}^2(t,X), \dots \bar{u}^N(t,X)\right)$ is a Nash equilibrium for the N coupled problems (3.2) if the following condition holds for each player, i.e. for $1 \le i \le N$

- The control $\bar{u}^i(t,X)$ is a solution to the optimal control problem for the i-th player with fixed $\left(u^1, \dots u^{i-1}, u^{i+1}, \dots u^N\right) = \left(\bar{u}^1, \dots \bar{u}^{i-1}, \bar{u}^{i+1}, \dots \bar{u}^N\right)$ i.e.

$$\begin{cases} \min_{u^i \in U} \int_0^\infty L^i \left(X(t), \bar{u}^1(t,X), \dots \bar{u}^{i-1}(t,X), u^i(t,X), \dots \bar{u}^N(t,X)\right) dt \\ \dot{x}^i(t) = f^i(x^i(t), \bar{u}^1(t,X), \dots u^i(t,X), \dots, \bar{u}^N(t,X), t) \qquad x^i(t_0^i) = x_0^i \end{cases} \tag{3.4}$$

  where $J^i\left[\bar{u}^1, \dots u^i, \dots \bar{u}^N, X_0, T_0\right] = \int_0^\infty L^i\left(X(t), \bar{u}^1, \dots u^i, \dots \bar{u}^N\right) dt$ is the cost function for each player i.

This definition of a feedback Nash equilibrium permits to consider the N-player differential game as N coupled optimal control problems. Hence, once again, the techniques we have shown for optimal control problems can be used to formulate sufficient conditions for Nash optimality. Let $\left(\bar{u}^1(t,X), \bar{u}^2(t,X), \dots \bar{u}^N(t,X)\right)$ be a Nash equilibrium and $\bar{X}(t) = \left((\bar{x}^1(t))^T, \dots (\bar{x}^i(t))^T, \dots \left(\bar{x}^N(t)\right)^T \right)^T$ the vector of the corresponding trajectories. We can define, for each player i, a value function $v^i(t,x)$ as follows

$$v^i(t_0^i, x_0^i) = J^i\left[\bar{u}^1, \dots \bar{u}^N, X_0, T_0\right] = \lim_{T\longrightarrow\infty}\int_0^T L^i\left(X(t), \bar{u}^1(t), \dots \bar{u}^N(t), t\right) dt$$

where the trajectory $\bar{x}^i(t)$ obeys the dynamic equation

$$\dot{\bar{x}}^i(t) = f^i(\bar{x}^i(t), \bar{u}^1(t, X), \dots, \bar{u}^N(t, X), t) \qquad \bar{x}^i(t_0^i) = x_0^i$$

If **(H1)** and **(H2)** hold, exploiting Theorem 2.12, we know that each value function has to satisfy the following system of N HJB equations

$$\begin{cases} v_t^1(t, x) + H\left(t, \bar{X}(t), v_x^1(t, x^1)\right) = 0 \\ \vdots \\ v_t^N(t, x) + H\left(t, \bar{X}(t), v_x^N(t, x^N)\right) = 0 \end{cases} \tag{3.5}$$

where, according to the conventions used in this work,
$v_x(t, x) = \frac{\partial v}{\partial x} = \left(\frac{\partial v}{\partial x_1}, \frac{\partial v}{\partial x_2} \ \cdots \ \frac{\partial v}{\partial x_n}\right) \in \mathbb{R}^{1\times d}$
$v_t(t, x) = \frac{\partial v}{\partial t} \in \mathbb{R}$
Substituting the Hamiltonian function computed in this case we have that (3.5) becomes

$$\begin{cases} v_t^1(t, x^1) - v_x^1(t, x^1) f^1\left(\bar{x}^1(t), \bar{u}^1, \dots, \bar{u}^N, t\right) - L^1\left(\bar{X}, \bar{u}^1, \dots \bar{u}^N, t\right) = 0 \\ \vdots \\ v_t^N(t, x^N) - v_x^N(t, x^N) f^N\left(\bar{x}^N(t), \bar{u}^1, \dots, \bar{u}^N, t\right) - L^N\left(\bar{X}, \bar{u}^1, \dots \bar{u}^N, t\right) = 0 \end{cases} \tag{3.6}$$

where the controls $\bar{u}^i$are given by

$$\bar{u}^i(t, X, v^i) = \arg\inf_{u^i\in U}\left\{v^i \cdot f^i(x^i, u^i, t) + L^i\left(X, \bar{u}^1, \dots \bar{u}^{i-1}, u^i, \bar{u}^{i+1}, \dots \bar{u}^N\right)\right\}$$

To sum up, the procedure that the Theorem 2.12 proposes in order to find a Nash equilibrium is

1. Compute the vector $\left(\bar{u}^1(t, X, v^1), \dots \bar{u}^N(t, X, v^N)\right)$ such as

   $$\bar{u}^i(t, X, v^i) = \bar{u}^i(t, X, v^i) = \arg\inf_{u^i\in U}\left\{v^i \cdot f^i(x^i, \bar{u}^1, \dots u^i, \dots \bar{u}^N, t) + L^i\left(X, \bar{u}^1, \dots, u^i, \dots \bar{u}^N\right)\right\}$$

   for each $1 \le i \le N$. Note that an infimum may not exist or, if it exists, it may not be unique.

2. Solve the system (3.6) and, if it has a solution $\left(\hat{v}^1(t,x^1),\ \dots\ \hat{v}^N(t,x^N)\right)$, the optimal control exists and is given by

$$\bar{u}^i(t,X,\hat{v}^i(t,x^i)) = \bar{u}^i(t,X)$$

for each $1 \le i \le N$.

In the closed-loop case we have some sufficient conditions and a procedure that permits to find at least one equilibrium, if it exists. Nevertheless, as in the case of optimal control, we need to solve a system of PDEs and an explicit solution is often impossible to find.

A way to simplify (3.6) is to consider only time-invariant value functions, i.e. $\left(v^1(t,x^1),\ \dots\ v^N(t,x^N)\right)$ such that

$$v^i(t_1,x^i) = v^i(t_2,x^i) \quad \forall t_1,t_2 > 0$$

for each $1 \le i \le N$. In order to do that it is sufficient to consider (3.6) where $v^i_t(t,x^i) = 0$ for each $1 \le i \le N$, namely

$$\begin{cases} v^1_x(x^1)\cdot f^1\left(\bar{x}^1(t),\bar{u}^1,t\right) + L^1\left(\bar{X},\bar{u}^1,\dots\bar{u}^1\right) = 0 \\ \vdots \\ v^N_x(x^N)\cdot f^N\left(\bar{x}^N(t),\bar{u}^N,t\right) + L^N\left(\bar{X},\bar{u}^N,\dots\bar{u}^N\right) = 0 \end{cases} \tag{3.7}$$

If at least one solution of (3.7) exists, the problem is referred to as *stationary.* In this case the time dependence is irrelevant and therefore the i-th player value function is simply indicated as $v^i(x^i)$. This will be one of the key concepts of the next chapter.

3.4 Approximate Solutions

We saw that finding a Nash equilibrium for a infinite horizon differential game is equivalent to solving the system of PDEs (3.6) or, in the stationary case, the system of PDEs (3.7). However, an explicit solution $\left(\bar{u}^1(t,X),\bar{u}^2(t,X),\ \dots\ \bar{u}^N(t,X)\right)$ can be computed only for a very narrow class of problems. For instance, the equilibrium for some simple linear quadratic differential games is explicitly found in [42] and some important results about uniqueness and existence of solutions are provided in [54]. In order to solve the more general and complex problems arising for example from biology, economics or social sciences it is necessary to study a way to construct approximate solutions. The purpose of this section is to provide information about the most recent studies about approximate solutions for differential games.

A significant amount of work has been done in the field of zero-sum games. For example an approximate equilibrium for some problems with direct practical applications to warfare and pursuit is provided in [25] and [29]. The technique adopted in these papers consists in linearizing the system dynamics around the origin and computing an explicit solution of the linearized problem. Only a small number of publications deal with nonzero-sum differential games. For example we can mention some remarks made in [36] and [39] about the extension of the procedures used for the optimal control problems.

A complete analysis about the Nash equilibria of a class of infinite horizon, nonzero-sum differential games is finally studied in [16]. In this paper the results of [15] are extended. In particular a strategy to find an approximate Nash equilibrium simply solving partial differential inequalities is proposed. These results will be also exploited in the following chapters in order to construct an approximate solution of mean field games.

Chapter 4

Stationary Mean Field Games

4.1 Introduction

The theory of mean field games is a branch of game theory and it is relatively recent. It was created in 2006 by J.-M. Lasry and P.-L. Lions and the first results and definitions were presented in [1]. Around the same time it was also developed independently by M. Huang, P.E. Caines and R.P. Malhame, see [2]. Advances in population dynamics were moreover made by Olivier Gulant, see [3]. Then, since 2009, some authors added further contributions or worked on new properties and applications of mean field games models.

A mean field game is a game that, as a differential game, can be described by a system of PDEs . However, in mean field games theory, three innovative key concepts are introduced

1. **infinitely many identical players** are considered and each one has a cost to minimize or maximize and can create their strategies based on the mean values of the other players' states;

2. the game is studied as **the interaction of each player with the rest of the group** and vice versa. In other words the whole system is defined by considering separately the influence of each individual on the mass of the other players and the influence of the mass of the other players on each individual;

3. players are not in total control of their strategies because of **an external noise** modeled as a Brownian motion.

Each of these new ideas find an application in different fields of science.

For example, the fact that mean field games deal with games with infinitely many identical players is exploited in game theory. As we have seen, differential games with N-players can be summed up by a system of HJB equations that often turns out to be very difficult to solve. Luckily, things become simpler, at least for a wide range of games that are symmetrical as far as players are concerned, when the number of players increases. Indeed, complex strategies can no longer be implemented by players because each player, in the eyes of other players, is progressively lost in the crowd when the number of players increases. Moving to the limit causes a situation in which each player becomes infinitesimal in the middle of the group of the other players. Therefore players build their strategies on the basis of their own state and the mean of the states of the infinite mass of the other players. They in turn create their strategies in the same way. For this reason, a complex system with mutual interaction between each couple of players can be simplified, using mean field games approach, considering only the interaction between a player and the mass of the others and vice versa.

This aspect is crucial, in particular, in particle physics. Indeed mean field theory represents a highly effective methodology for handling a wide variety of situations in which there is a large number of particles. In such cases the dynamics can no longer be described by modeling all the inter-particle interactions. Therefore it is very useful to introduce the concept of *mean field*. This latter is a statistical description of the behavior of a set made by a huge number of elements. Using this notion it is possible to construct an approximation of the situation. The mean field plays the role of a mediator between a single particle and rest of the system. In other words, the mean field games theory allows to separately describe the contribution of each particle to the creation of a mean field and the effect of the mean field on each particle, see, for example, [49].

Mean field games theory is also used in economics where, on the contrary, agents usually have little concern about each other because everyone looks only to his own interest and to market prices. Indeed this theory can be exploited to model external phenomena with statistical nature that are very common in the economic framework. Therefore most mean field games models are used not only to describe but also, and more importantly, to explain a phenomenon and hence to predict future developments. The latter is a crucial aspect in economics, see, for instance, [46].

Mean field applications are many and diverse, see, for instance, [52] and [50] where crown dynamics and social interactions are studied or [5] where the opinions making process in analyzed.

In this chapter, a formal description of mean field games, a definition of what is usually meant by mean field game solution and a way to construct it will be provided. All the concepts used to explain optimal control problems and differential games will be reused. Moreover some new concepts will be introduced to model the three previously described key concepts of mean field games. The concept of stationary solution will be investigated again and its interpretation will be provided.

In the last section of the chapter, an approximate solution of a class of mean field games will be constructed. A proof of the accuracy of the proposed approximate solution will be presented and its effectiveness will be shown in a numerical example. We remark that, in this last section, a very innovative topic is proposed because there are very few publications dealing with approximation problems for mean field games, see for example [51]. Moreover some explicit solutions are only provided for a very narrow class of mean field games such as the linear quadratic ones in [4] and [6]. Even if the approach is the same used in [15] and [16], several new ideas are here introduced in order to manage the deeply different structure of the involved equations.

4.2 Definition

We start by introducing the definition of Brownian motion

Definition 4.1. [47](Brownian motion) A Brownian motion is a set of random variables indexed by time t denoted by $W(t)$ and such that

- $W(0) = 0$ with probability 1;
- the function $t \longmapsto W(t)$ is continuous on $[0, T]$ for each $T > 0$;

- for $0 < s < t$ the increment $W(t) - W(s)$ is a random normal variable of mean 0 and variance $t - s$ namely $W(t) - W(s) \sim \mathcal{N}(0, t - s)$;
- for $0 < s < t < u < v$ the increments $W(t) - W(s)$ and $W(v) - W(u)$ are independent.

Extending the definition provided in [1] we define a nonlinear mean field game as follows.

Definition 4.2. [1](Mean field game) A mean field game consists of the following elements

- an infinite set of players P, where $|P| = \infty$, and such that there exists a bijective correspondence between P and the real numbers set $\mathbb{R}$. In this way each real number identifies a player;
- a constant $t_0 \geq 0$ that represents the time instant at which we assume to know the initial conditions of the problem;
- the set $\mathcal{X}$ of the bounded smooth functions $x(t) : [0, \infty[\longmapsto \mathbb{R}^d$ where d is the dimension of the state;
- a given function $m_0(x) : \mathbb{R}^d \longmapsto [0, +\infty[$, that will be referred to as *initial population density function*, satisfying the following condition

$$\int_{\mathbb{R}^d} m_0(x) dx_0 = 1$$

  where $m_0(x) > 0$ for each $x \in \mathbb{R}^d$ and, for each player, $m_0(\bar{x}_0)dx$ is the probability that the initial state at t_0 of such player is in the infinitesimal range $[\bar{x}_0, \bar{x}_0 + dx_0]$;
- a given function $m(x, t) : \mathbb{R}^d \longmapsto [0, +\infty[$, that will be referred to as *population density function*, such that

$$\int_{\mathbb{R}^d} m(x, t) dx = 1 \quad \forall t \geq 0$$

  where $m(x) > 0$ for each $x \in \mathbb{R}^d$. For each player, $m(\bar{x})dx$ is the probability that the state of such a player is in the infinitesimal range $[\bar{x}, \bar{x} + dx]$ at time $t \geq 0$. Because of the definition of initial population density function we have that $m_0(x) = m(x, t_0)$ for each $x \in \mathbb{R}^d$;

- the set U_x of the bounded smooth functions $u(x,m,t) : \mathbb{R}^d \times [0,\infty[\times[0,\infty[\longmapsto \mathbb{R}^b$ where b is the dimension of the control input;

- a d-dimension standard Brownian motion $W(t) = (W_1(t), W_2(t) \quad \dots \quad W_d(t))^T$ where $W_i(t)$ and $W_j(t)$ are Brownian motions and they are independent of each other for each $i \neq j$;

- an initial condition $x_0 \in \mathbb{R}^d$ that is a realization of the stochastic variable that has $m_0(x)$ as probability density function;

- a known dynamic equation, that is the same for all players, given by

$$\dot{x}(t) = f(x(t), u(x,m,t), t) + \sigma\dot{W}(t) \quad x(t_0) = x_0 \tag{4.1}$$

  that describes how each player's state changes as a result of the input $u(x,m,t)$. We have that $f(x,u,t) : \mathbb{R}^d \times \mathbb{R}^b \times [0,\infty[\longmapsto \mathbb{R}^d$ is smooth and $\sigma \in \mathbb{R}^{d\times d}$ is a matrix with $det\,(\sigma) \neq 0$;

- a known cost function, that is the same for each player, given by

$$\begin{aligned} &J\left(x_0, u(x,m(x),t), x(t), m(x)\right) := \\ &\lim_{T\longrightarrow\infty} E \int_0^T L\left(u(x,m,t), x(t), m(x), t\right) dt \end{aligned} \tag{4.2}$$

  where $L\left(u(x,m(x),t), x(t), m(x), t\right) : \mathbb{R}^b \times \mathbb{R}^d \times [0,\infty[\times[0,\infty[\longmapsto \mathbb{R}$ is smooth;

- the problem

$$\begin{cases} u(x,m,t) = \arg\min_{u(x,m,t)\in U_x} \left(E \int_0^\infty L\left(u(x,m,t), x(t), m(x,t), t\right) dt\right) \\ \dot{x}(t) = f(x(t), u(x,m,t), t) + \sigma(x)W(t) \\ \int_{\mathbb{R}^d} m(x)dx = 1 \quad \forall t \geq 0 \\ x(t_0) = x_0 \\ m(x,t_0) = m_0(x) \end{cases} \tag{4.3}$$

  that has to be solved by choosing a proper feedback control $u(x,m,t)$ and consequently a population distribution $m(x,t)$.

Remark 4.3. In the mean field games framework only feedback control laws of the form $u(x, m, t)$ with an explicit dependence on x are considered. This aspect is consistent with the idea of partitioning the whole system into two sets: a set with a single player and another set with the remaining players. Indeed, in this way, each player can choose his input according to his own state x and the statistical information m about the other players' states but can not use information about the state of a particular other player.

Note that every player is equal i.e. each player has the same dynamics and the same cost function. For this reason the players will be referred to as *identical*. Moreover a feedback solution $\bar{u}(x, m, t)$ that can be found by solving problem (4.3) is obviously the same for each player. This is the reason why in the definition and in (4.3), differently from the differential games framework, the symbols $u^i(x, m, t)$ and $x^i(t)$ with the i apex are not used. Because of what we have just explained, in the mean field game problems the concept of optimal solution that we have introduced in the Chapter 2 and the concept of Nash equilibrium that we have described in the Chapter 3 are the same. Indeed the conditions (2.8) and (3.4) are equivalent in case identical players are considered. From now on we will use the words solution, equilibrium or optimum with the same meaning intending the pair of smooth functions $(u(x, m, t), m(x))$ that solve the system (4.3).

Remark 4.4. As we said, each player has the same dynamic equation and the same control input is used. However, considering two players $r, s \in \mathbb{R}$ with $r \neq s$, it is not true that $x^r(t) = x^s(t)\ \forall t \geq 0$. Indeed, each player's state is generally different from the others due to

- *initial conditions* that are governed by $m_0(x)$ and therefore are generally different for each player;
- *Brownian motion* that influences the dynamics and is independent for each player.

4.3 HJB and FPK Equations

The equilibrium of a mean field game is given by the solution of the system (4.3). It can be thought of as an optimization problem where two constraints are present. The first one is the differential equation that describes the dynamics of each player and the second one is given by the fact that the integral of the population density function $m(x,t)$ in $\mathbb{R}^d$ has to be equal to 1, because it is a probability density function. Paradoxically, the fact that we are considering infinitely many players described by a single function $m(x,t)$ allows us to consider the mean field game as a single player optimal control problem. Indeed a value function $v(x,m,t) : \mathbb{R}^d \times [0,\infty[\times[0,\infty[\longmapsto \mathbb{R}$ can be defined as follows

$$v(\hat{x}_0, \hat{m}_0(x), \hat{t}_0) = \inf_{u(x,m)\in U_x} \lim_{T\longrightarrow\infty} E \int_0^T L\left(\bar{u}(x,m), \bar{x}(t), \bar{m}(x,t), t\right) dt \quad (4.4)$$

where

- $(\bar{u}(x,m), \bar{m}(x,t))$ is the optimal solution of (4.3) with initial conditions $m(x,\hat{t}_0) = \hat{m}_0(x)$ and $x(\hat{t}_0) = \hat{x}_0$;
- $\bar{x}(t)$ is the solution of (4.1) corresponding to the input $\bar{u}(x,m)$ with $x(0) = \hat{x}_0$. It is a stochastic process because of the Brownian motion but, in (4.4), we are considering its expectation that is deterministic.

As in the optimal control problems and differential games, there exists a system of PDEs to solve in order to find the optimal solution, namely the couple $(u(x,m,t), m(x,t))$. In this case the system is the following one

$$\begin{cases} -v_t(x,t) - tr(\nu v_{xx}(x,m,t)) + H(x, v_x(x,m,t), m) = 0 \\ -m_t(x,t) - tr(\nu m_{xx}(x,t)) - div\left(\frac{\partial H(x,v_x(x,m,t),m)}{\partial p} m(x,t)\right) = 0 \end{cases} \quad (4.5)$$

where

- $div(\cdot)$ and $tr(\cdot)$ are respectively the divergence and the trace operator;
- $H(x,p,m)$ is the minimized Hamiltonian defined by (2.9) where m is regarded as constant;
- $v(x,m)$ is the value function described by (4.4);
- v_x, v_{xx}, m_x, m_{xx} are the gradients and the Hessian matrices of $v(x)$ and $m(x)$, respectively. Note that gradients are considered row vectors according to the convention adopted in this whole thesis;

- $\nu \in \mathbb{R}^{d\times d}$ is a matrix and it can be calculated by the expression $\nu = \frac{\sigma\sigma^T}{2}$.

Moreover the optimal control $u(x, m, t)$, as in the singular player case and according to (2.15), is given by

$$u(t, m, x) = \arg\min_{u(t)\in U} \left(v_x(t, m, x) f(x, u, t) + L(u, x, m, t)\right) \qquad (4.6)$$

Remark 4.5. The first PDE of (4.5) is a stochastic HJB equation that differs from (2.14) because of the term $tr(\nu v_{xx}(x, m, t))$ that is due to the presence of the Brownian motion. This PDE is linked not only to the cost function and to the dynamics but also to $m(x, t)$ and is used to find the optimal control input for each player. Roughly speaking it describes how each player's behavior is influenced by the other players' behavior that is modeled by $m(x, t)$.

Remark 4.6. The second PDE of (4.5) is referred to as Fokker-Planck-Kolmogorov (FPK) equation. It describes the way in which each player's control input $u(x, m, t)$ influences the density function $m(x, t)$. Roughly speaking it describes how the behavior of the mass of players is influenced by each player's behavior.

The formal derivation of the equations requires concepts of stochastic differential games and diffusion processes that we will not deal with in this work. We simply note that the derivation of HJB equation is similar to the procedure that we have shown in Theorem 2.12. The only difference is the fact that now we have to consider the presence of the Brownian motion and hence we have the additional term $tr(\nu v_{xx}(x, m, t))$. For a full explanation of the derivation of the HJB stochastic equation see [48]. While for further informations about the FPK equation see [47].

In the literature the PDE (4.5) alone, due to a language misuse, is often referred to as mean field game and the other details are neglected. The solution of (4.5) is often difficult or impossible to explicitly compute. For this reason, in particular applications, numerical techniques are used as explained in [43]. Moreover no conditions for the existence or the uniqueness of the solution are currently available in literature in the general case. For this reason currently we can only say that, if a pair $(u(x, m, t), m(x, t))$ satisfies (4.5) it is an equilibrium for the problem defined in Definition 4.2.

In the mean field game framework the concept of stationary solution can be explained by taking into account the three following cases

1. *stationary w.r.t. control solution* $(u(x,m), m(x,t))$ that is an equilibrium where the control $u(x,m)$ does not depend explicitly on time. This means that the value function is the same for each initial instant t_0 when we know the initial conditions x_0 and m_0 namely

$$v(x_0, m_0, t_1) = v(x_0, m_0, t_2) \quad \forall t_1, t_2 \geq 0$$

This implies that $v_t(x,t) = 0$ and consequently the HJB equation becomes

$$-tr(\nu v_{xx}(x,m,t)) + H(x, v_x(x,m,t), m) = 0$$

Note that, even if the optimal control depends only on the player's state x and the "system state" m, the population density function can change over time.

2. *stationary w.r.t. population density function solution* $(u(x,m,t), m(x))$ that is an equilibrium where the population density function $m(x)$ does not depend explicitly on time. This means that the density function that describes the other players' state is always the same namely

$$m(x, t_1) = m(x, t_2) \quad \forall t_1, t_2 \geq 0$$

Note that this does not mean that each player's state remains forever the same, but simply that their motion is such that their distribution remains constant. Such a motion can be generated by a control $u(x,m,t)$ explicitly depending on time. Moreover note that, in this case, the initial population distribution $m_0(x)$ may be unknown. Indeed such a stationary solution $(u(x,m,t), m(x))$ is meant to be the approximation of the behavior of a mean field game after a very long time when the contribution of the initial condition $m_0(x)$ has disappeared and $m(x,t)$ is almost constant in time. Such a situation implies that $m_t(x,t) = 0$ and, consequently, the FPK equation becomes

$$-tr(\nu m_{xx}(x,t)) - div\left(\frac{\partial H(x, v_x(x,m,t), m)}{\partial p} m(x,t)\right) = 0$$

In the case that we want to impose an initial condition $m_0(x)$ it follows that $m(x,t) = m_0(x)$ for each $t > 0$. Therefore the stationary solution is $(u(x,m,t), m_0(x))$ where $u(x,m,t)$ is a control input that minimizes the cost function (4.2) maintaining a constant population density.

3. *stationary both w.r.t. population density function and w.r.t. control solution* $(u(x,m), m(x))$ that is an equilibrium where both the control $u(x,m)$ and the population density function $m(x)$ do not depend explicitly on time. In this situation we have both the previously explained phenomena and consequently (4.5) reads

$$\begin{cases} -tr(\nu v_{xx}(x,m,t)) + H(x, v_x(x,m,t), m) = 0 \\ -tr(\nu m_{xx}(x,t)) - div\left(\frac{\partial H(x,v_x(x,m,t),m)}{\partial p} m(x,t)\right) = 0 \end{cases} \tag{4.7}$$

In the literature a mean field game described by the system (4.5) where $v_t(x,t) = 0$ and consequently admitting only *stationary w.r.t. control solution* $(u(x,m), m(x,t))$ is referred to as *stationary w.r.t. control mean field game*. The same holds for the other two stationary cases in particular a *stationary both w.r.t. population density function and w.r.t. control mean field game* is simply referred to as *stationary mean field game*. The latter is the focus of this work and an approach for constructing approximate solutions for such problems will be provided in the following sections.

Finally we remark that, in mean field games, discounted or average cost functions are almost always used instead of (4.2). They are expressed as

$$J_{disc}\left(x_0, u, x, m_0\right) := \lim_{T \longrightarrow \infty} E \int_0^T e^{-\alpha t} L\left(u(x,m,t), x(t), m(x)\right) dt \tag{4.8}$$

and

$$J_{ave}\left(x_0, u, x, m_0\right) := \lim_{T \longrightarrow \infty} \frac{1}{T} E \int_0^T L\left(u(x,m,t), x(t), m(x)\right) dt \tag{4.9}$$

respectively. Consequently the correspondent HJB equations become

$$\alpha v(x,t) - v_t(x,t) - tr(\nu v_{xx}(x,m,t)) + H(x, v_x(x,m,t), m) = 0 \tag{4.10}$$

in the discounted cost function case, where $\alpha > 0$ is the same constant appearing in (4.8), and

$$\lambda - v_t(x,t) - tr(\nu v_{xx}(x,m,t)) + H(x, v_x(x,m,t), m) = 0 \tag{4.11}$$

in the average cost function case where λ is an unknown constant to be determined.

Short proofs for the derivation of HJB PDEs with discounted and average cost function are available in [44] and in [45], respectively. Note that, if we use the average cost, the solution of the corresponding system of HJB FPK PDEs is the triple $(u(x,m,t), m(x,t), \lambda)$.

4.4 Proposed Approximate Solution

4.4.1 Stationary Problem

In this section we want to provide a **dynamic** method for constructing a **local approximate** solution of a specific class of **stationary** mean field games. This class is chosen firstly because (4.7) is simpler than (4.5) to solve and secondly because we want to exploit the results provided in [15] where an explicit solution of a stationary linear quadratic mean field game is computed. Our aim is to use those results to design a local solution of a nonlinear stationary mean field game that, if linearized around the origin, becomes exactly like the one studied in [15]. Moreover the main strength of the approximate equilibrium that we want to propose is the fact that it can be found by simply solving algebraic inequalities instead of PDEs.

The class of stationary mean field games that we want to study is characterized by the following dynamic equation for each player

$$dx(t) = [f(x(t)) + g(x(t))u(x)]\,dt + \sigma dW(t) \quad x(0) = x_0$$

or equivalently

$$\dot{x} = f(x) + g(x)u(x) + \sigma\dot{W} \quad x(0) = x_0 \tag{4.12}$$

where

- d is the dimension of the state;
- b is the dimension of the input;
- $x(t):\ [0,\infty[\longmapsto \mathbb{R}^d$ is the state of the player that we are considering and it is an element of the states set $\mathcal{X}$;
- $u(x):\ \mathbb{R}^d \longmapsto \mathbb{R}^b$ represents the control and that we have to choose and it is an element of the bounded smooth functions set U_x;
- $x_0 \in \mathbb{R}^d$ is the initial state;
- $f(x):\ \mathbb{R}^d \longmapsto \mathbb{R}^d$ is a C^2 class function or, in other words, is a function that can be differentiated at least twice. It is given and it represents,

roughly speaking, how the state behaves in the absence of input signals. Moreover $f(x)$ is such that $f(0) = 0$ therefore there exists some, possibly not unique, smooth function $F(x) : \mathbb{R}^d \longmapsto \mathbb{R}^{d\times d}$ such that $f(x) = F(x)x$ for all x;

- $g(x) : \mathbb{R}^d \longmapsto \mathbb{R}^{d\times b}$ is another C^2 class function and describes the way in which the state $x(t)$ is affected by the control $u(x)$;
- $\sigma \in \mathbb{R}^{d\times d}$ is a matrix such that $\det(\sigma) \neq 0$ and $tr(\sigma) > 0$. It quantifies the noise contribution;
- $W(t) = (W_1(t), W_2(t) \ \ \dots \ \ W_d(t))^T$ is the d-dimension standard Brownian motion where $W_i(t)$ and $W_j(t)$ are independent for each $i \neq j$.

Finally, as we said, the *population distribution function* $m(x)$ is $m(x) : \mathbb{R}^d \longmapsto [0, +\infty[$ and, since it is a probability density function, it holds that

$$\int_{\mathbb{R}^d} m(x)dx = 1 \tag{4.13}$$

We introduce the average cost function, of the kind shown in (4.9), as follows

$$J\left(x_0, u(x), m(x)\right) := \lim_{T\to\infty} \frac{1}{T} E\left[\int_0^T \left(\frac{\left(u(x)\right)^T E\, u(x)}{2} + V\left[m\right]\left(x\right)\right) dt\right] \tag{4.14}$$

where

- $E \in \mathbb{R}^{b\times b}$ is a symmetric and positive definite matrix;
- $V\left[m\right]\left(x\right)$ is a functional that describes the part of the cost function depending on the population distribution $m(x)$.

In this work we define the functional $V[m](x)$ as

$$V[m](x) = q(x(t)) - d(x)\ln\left(m(x(t))\right)$$

where

- $q(x) : \mathbb{R}^d \longmapsto \mathbb{R}$ is a smooth function expressed as $q(x) = x^T Q(x)x$ for each $x \in \mathbb{R}^d$;

- $Q(x) : \mathbb{R}^d \longmapsto \mathbb{R}^{d\times d}$ is a positive semi-definite smooth function, in other word it is a function such that $Q(x) \geq 0$ for each $x \in \mathbb{R}^d$;
- $d(x) : \mathbb{R}^d \longmapsto \mathbb{R}$ is a smooth function such that $d(x) > 0$ for each $x \in \mathbb{R}^d$.

Remark 4.7. The cost function that we have introduced aims to minimize the norm of the state $x(t)$ and enforces the fact that each player has the same state value. Indeed $Q(x)$ is positive definite and hence it penalizes states with a large norm. Then the term $-\ln(m(x(t)))$ is used, loosely speaking, in order to force the state to avoid zones with small density of players and to move into denser areas. In other words, by minimizing $-\ln(m(x(t)))$, we are choosing states $\hat{x}$ such that $m(\hat{x})$ is large, namely states such that it is very probable to find another player with a similar value of the state.

In summary, the proposed mean field game problem is described by the following system

$$\begin{cases} \dot{x} = f(x) + g(x)u(t) + \sigma\dot{W} \qquad x(0) = x_0 \\ J\left(x_0, u(x), m(x)\right) := \lim_{T\to\infty} \frac{1}{T}\, E\left[\int_0^T \left(\frac{(u(x(t)))^T E\, u(x(t))}{2} + V\left[m\right]\left(x(t)\right)\right) dt\right] \\ u(x) = \arg\min_{u(x)\in U_x}\left(J\left(x_0, u(x), m(x)\right)\right) \\ \int_{\mathbb{R}^d} m(x)dx = 1 \end{cases} \tag{4.15}$$

and we want to find, if it exists, a pair $(\bar{u}(x), \bar{m}(x))$ that solves it. As we are in the stationary case such pair $(\bar{u}(x), \bar{m}(x))$, as it is shown in [16], is also such that

$$\begin{aligned} &E\left(\frac{(\bar{u}(\bar{x}(t_1)))^T E\, \bar{u}(\bar{x}(t_1))}{2} + V\left[m\right]\left(\bar{x}(t_1)\right)\right) \geq \\ &E\left(\frac{(\bar{u}(\bar{x}(t_2)))^T E\, \bar{u}(\bar{x}(t_2))}{2} + V\left[m\right]\left(\bar{x}(t_2)\right)\right) \end{aligned} \tag{4.16}$$

for all $0 < t_1 < t_2$ where $\bar{x}(t)$ is the solution of (4.12) corresponding to the input $\bar{u}(x)$ and to the initial condition x_0.

4.4.2 Hamiltonian Computation

As we explained in the previous section, in order to define the HJB and the FPK equations that allow to find the optimal $u(t)$ and $m(x)$, it is necessary to compute the Hamiltonian function. The latter, according to the definition (2.9), is computed as follows

$$H(x,p,m) := \inf_{u\in U_x}\left\{+u^T\frac{E}{2}u + V[m](x) + p\left(f(x)+g(x)u\right)\right\} \tag{4.17}$$

where $p \in \mathbb{R}^{1\times d}$ is a row vector which plays the role of the independent variable. Noting that some terms do not depend on u the previous expression becomes

$$H(x,p,m) = +pf(x) + V[m](x) + \inf_{u\in U_x}\left\{+u^T\frac{E}{2}u + pg(x)u\right\}$$

Under the assumption that $-u^T\frac{H}{2}u+pg(x)u$ is convex in u and that it admits a minimum, such minimum can be found calculating the first derivative of $-u^T\frac{H}{2}u + pg(x)u$ as follows

$$\frac{\partial\left(+u^T\frac{H}{2}u + pg(x)u\right)}{\partial u} = +u^TE + pg(x) = 0$$

Hence the minimum u_{min} is

$$u_{min}^T = -pg(x)E^{-1} \;\Rightarrow\; u_{min} = -E^{-1}g^T(x)p^T \tag{4.18}$$

By substituting the computed u_{min} in the equation (4.17) we obtain

$$\begin{aligned} H(x,p,m) &= +p\,f(x) + V[m](x) + \frac{1}{2}pg(x)E^{-1}EE^{-1}g^T(x)p^T \\ &\quad -pg(x)E^{-1}g^T(x)p^T \\ &= +p\,f(x) + V[m](x) - \frac{1}{2}pg(x)E^{-1}g^T(x)p^T \end{aligned} \tag{4.19}$$

4.4.3 HJB and FPK Equations

According to what we have explained in Section 4.3, the problem (4.15) can be solved considering the system made coupling an HJB equation with a FPK equation. Such equations are given by (4.7) where the first equation is substituted by (4.11) because we are considering an average cost function. In other words the system of PDEs that has to be solved is the following one

$$\begin{cases} -tr(\nu v_{xx}(x)) + H(x, v_x(x), m) + \lambda = 0 \\ -tr(\nu m_{xx}(x)) - div\left(\frac{\partial H(x,v_x(x),m)}{\partial p} m(x)\right) = 0 \end{cases} \tag{4.20}$$

The partial derivative of the Hamiltonian function $H(x, p, m)$ (4.19) is

$$\frac{\partial H(x,p,m)}{\partial p} = f^T(x) - pg(x)E^{-1}g^T(x) \tag{4.21}$$

Then, by substituting (4.21) and (4.19) in (4.20) and by dropping the dependence on the state x, we have

$$\begin{cases} -tr(\nu v_{xx}) + v_x\, f - \frac{1}{2} v_x g E^{-1} g^T v_x^T + q - d \ln(m) + \lambda = 0 \\ -tr(\nu m_{xx}) - div\left(+m\, f^T - m\, v_x g E^{-1} g^T\right) = 0 \end{cases} \tag{4.22}$$

and we have to find the triple $(m(x), v(x), \lambda)$. It is also known from the optimal control theory in Chapter 2 that, given $v(x)$, the optimal feedback control $u_{opt}(x)$ is given by (4.18) namely

$$u_{opt}(x) = -E^{-1}g^T(x)v_x(x)^T \tag{4.23}$$

Moreover, according to [45], in the stationary case the constant λ is such that

$$\lambda = \inf_{x_0 \in \mathbb{R}^d} J\left(x_0, u_{opt}(x)\right) \tag{4.24}$$

$$= \inf_{x_0 \in \mathbb{R}^d} \lim_{T\to\infty} \frac{1}{T} E\left[\int_0^T \left(\frac{(u_{opt}(x))^T E\, u_{opt}(x)}{2} + V\left[m\right](x(t))\right) dt\right] \tag{4.25}$$

Remark 4.8. We note that, since the term $-d(x)\ln(m(x(t)))$ in $V[m](x(t))$ is negative, λ may be negative.

In general (4.22) is easily solvable only in a very small number of cases studied for instance in [4] and [6]. One of them, namely the linear-quadratic case, will be dealt with in the following section. Moreover we will define three problems, different from this one, that will be used to explain what we mean by **local approximate dynamic** solution of (4.15).

4.4.4 Linear-Quadratic Case

An explicit solution $(m(x), v(x), \lambda)$ of the system (4.22) and consequently of the mean field game (4.15) has been computed only in the linear-quadratic case (see [4]). The procedure to compute the optimum in this specific case is briefly provided because it will be helpful to understand the approach used in the following.

For the linear-quadratic problem we have that

$F(x) \equiv A \in \mathbb{R}^{d \times d}$

$g(x) \equiv B \in \mathbb{R}^{d \times b}$

$Q(x) \equiv Q \in \mathbb{R}^{d \times d}$

$d(x) \equiv d \in \mathbb{R}$

The dynamics (4.12) becomes

$$\dot{x} = Ax + Bu(t) + \sigma \dot{W} \tag{4.26}$$

and, similarly, the cost function (4.14) becomes

$$J\left(x_0, u, m\right) := \lim_{T \to \infty} \frac{1}{T} \, E\left[\int_0^T \left(\frac{u(x)^T E \, u(x)}{2} + x^T Q x - d \ln\left(m(x)\right) \right) dt \right] \tag{4.27}$$

So, the coupled HJB and FPK equations are

$$\begin{cases} -tr(\nu v_{xx}) + v_x \, Ax - \frac{1}{2} v_x B E^{-1} B^T v_x^T + \lambda = -x^T Q \, x + d \ln\left(m\right) \\ -tr(\nu m_{xx}) - div\left(+m \, x^T A - m \, v_x B E^{-1} B^T \right) = 0 \end{cases} \tag{4.28}$$

Theorem 4.9. *[4]A solution $(m(x), v(x), \lambda)$ of (4.28) always exists and it is expressed as*

$$v(x) = \frac{1}{2} x^T P x \tag{4.29}$$

$$m(x) = \frac{1}{\left(\sqrt{2\pi}\right)^d \left(det(G)\right)^{-\frac{1}{2}}} \, e^{-\frac{1}{2} x^T G x} = \kappa e^{-\frac{1}{2} x^T G x} \sim \mathcal{N}\left(0, G^{-1}\right) \tag{4.30}$$

$$\lambda = tr\left(P\nu\right) + d\ln k \tag{4.31}$$

where $P, G \in \mathbb{R}^{d\times d}$ *are symmetric,* G *is positive definite and they solve the system*

$$\begin{cases} PBE^{-1}B^T - A + G\nu = 0 \\ -PBE^{-1}B^TP + A^TP + PA + 2Q + dG = 0 \end{cases} \tag{4.32}$$

Hence, the optimal control is expressed as

$$u_{opt}(x) = -E^{-1}B^TPx$$

Proof. A brief proof of the correctness of the solution is provided. Compute first v_x and m_x exploiting the derivation rules (A.2) and (A.3) available in Appendix A as follows

$v_x(x) = x^TP$

$m_x(x) = -m(x)\left(x^TG\right)$

Exploiting the property of the trace operator (A.10) available in Appendix A and substituting (4.30) in the second equation of (4.28), namely the FPK equation, we have

$$\begin{aligned} 0 &= -tr(\nu m_{xx}) - div\left(+m\,x^TA - m\,v_xBE^{-1}B^T\right) \\ &= -div\left(m_x\nu + m\,x^TA - m\,v_xBE^{-1}B^T\right) \\ &= div\left(m\left(-x^TG\nu + x^TA - v_xBE^{-1}B^T\right)\right) \end{aligned}$$

Then, substituting the proposed value function (4.29) in the previous expression we obtain

$$0 = div\left(m\left(-x^TG\nu + x^TA - x^TPBE^{-1}B^T\right)\right) \tag{4.33}$$

Exploiting the property of the divergence operator (A.1) explained in Appendix A and remembering that $m(x) > 0$ for each $x \in \mathbb{R}^d$ (and hence in particular $m(x) \neq 0$) we have for each $x \in \mathbb{R}^d$

$$
\begin{aligned}
0 &= div\left(m\left(-x^T G\nu + x^T A - x^T PBE^{-1}B^T\right)\right) \\
&= \left(-x^T G\nu + x^T A - x^T PBE^{-1}B^T\right) m_x^T \\
&\quad + m\, div\left(-x^T G\nu + x^T A - x^T PBE^{-1}B^T\right) \\
&= -m\left(-x^T G\nu + x^T A - x^T PBE^{-1}B^T\right) Gx \\
&\quad + m\, tr\left(-G\nu + A - PBE^{-1}B^T\right) \\
&= x^T G\nu Gx - x^T AGx + x^T PBE^{-1}B^T Gx \\
&\quad + tr\left(-G\nu + A - PBE^{-1}B^T\right)
\end{aligned}
$$

This implies

$$
\begin{cases}
G\nu G - x^T AG + x^T PBE^{-1}B^T G = 0 \\
-G\nu + A - PBE^{-1}B^T = 0
\end{cases}
$$

that can be equivalently rewritten as

$$
+G\nu - A + PBE^{-1}B^T = 0
$$

which is the first condition of (4.32). Substituting the proposed solutions (4.29), (4.30) and (4.31) in the first equation of (4.28), namely the HJB equation, we obtain

$$\begin{aligned} 0 &= -tr(\nu v_{xx}) + v_x\, Ax - \frac{1}{2} v_x B E^{-1} B^T v_x^T + \lambda \\ &\quad + x^T Q\, x - d \ln(m) \qquad (4.34) \\ &= -tr(\nu P) + x^T P A x - \frac{1}{2} x^T P B E^{-1} B^T P x \\ &\quad + tr\,(P \nu) + d \ln k + x^T Q\, x \\ &\quad - d \ln k + \frac{1}{2} d x^T G x \\ &= x^T P A x - \frac{1}{2} x^T P B E^{-1} B^T P x + x^T Q\, x + \frac{1}{2} d x^T G x \qquad (4.35) \end{aligned}$$

Finally note that, since $x^T P A x$ is a scalar term,

$$x^T P A x = x^T A^T P x$$

and in particular the following holds

$$x^T P A x = \frac{1}{2} x^T P A x + \frac{1}{2} x^T A^T P x$$

As a consequence, (4.35) can be rewritten as follows

$$\begin{aligned} 0 &= \frac{1}{2} x^T P A x + \frac{1}{2} x^T A^T P x - \frac{1}{2} x^T P B E^{-1} B^T P x \\ &\quad + x^T Q\, x + \frac{1}{2} d x^T G x \end{aligned}$$

Once again the last expression has to be true for each $x \in \mathbb{R}^d$, which implies

$$PA + A^T P - P B E^{-1} B^T P + 2Q + dG = 0$$

that is the second condition of (4.32).

All described steps are true for each $x \in \mathbb{R}^d$ and therefore the proposed optimal solution is global. □

Remark 4.10. If we consider a linear-quadratic problem with $d = 0$ in the cost function (4.27), the second equation of (4.32) becomes the Riccati equation and it does not contain the term G. For this reason it can be solved in P. This means that P and hence $u(x)$ can be found without knowing the value of G and thus of $m(x)$. This fact can be explained noting that, when $d = 0$, the cost function becomes

$$J\left(x_0, u(x)\right) := \lim_{T\to\infty} \frac{1}{T}\, E\left[\int_0^T \left(\frac{\left(u(x)\right)^T E\, u(x)}{2} + x^T Q x\right) dt\right]$$

where no terms depending on $m(x)$ appears.

4.4.5 Problem 1: Local optimum problem

Since (4.22) is in general very difficult to be solved in $\mathbb{R}^d$ we want to consider a **local** optimal solution. For this reason we define a new problem that differs from (4.22) only because the optimal control $u(x)$ is defined as

$$u(x) = \arg\min_{u(x)\in U_{\bar{\Omega}}} \left(J\left(x_0, u(x), m(x)\right)\right) \tag{4.36}$$

where

$$U_{\bar{\Omega}} = \left\{u(x) : \bar{\Omega} \longmapsto \mathbb{R}^b \text{ with } u(x) \text{ smooth }\right\}$$

and $\bar{\Omega}$ is a non-empty neighborhood of the origin. Such problem will be referred to as local optimum problem or Problem 1 and it is fully described by the following system

$$\begin{cases} \dot{x} = f(x) + g(x)u(t) + \sigma\dot{W} \qquad x(0) = x_0 \\ J\left(x_0, u(x), m(x)\right) := \lim_{T\to\infty} \frac{1}{T}\, E\left[\int_0^T \left(\frac{\left(u(x(t))\right)^T E\, u(x(t))}{2} + V\left[m\right]\left(x(t)\right)\right) dt\right] \\ u(x) = \arg\min_{u(x)\in U_{\bar{\Omega}}} \left(J\left(x_0, u(x), m(x)\right)\right) \\ \int_{\mathbb{R}^d} m(x)dx = 1 \end{cases} \tag{4.37}$$

The solution $(u(x), m(x))$ of (4.37) will be referred to as **local** solution of (4.22).

Remark 4.11. Note that we are interested only in the control $u(x)$ that minimizes the cost function in $\bar{\Omega}$. For this reason, if a function $\tilde{u}(x) : \mathbb{R}^d \longmapsto \mathbb{R}^b$ is such that (4.36) holds, then infinitely many functions $\check{u}(x) : \mathbb{R}^d \longmapsto \mathbb{R}^b$ such that (4.36) holds exist. Indeed a function $\check{u}(x)$ is such that (4.36) holds if $\check{u}(x) = \tilde{u}(x)$ for each $x \in \bar{\Omega}$.

Remark 4.12. In this framework $m(x)$ is still a probability density function but it can be used to determine the probability that a player is in the state x only if $x \in \bar{\Omega}$.

The local optimum problem can be solved finding the solution of the system (4.22) in a neighborhood of the origin. In other words, we need to find a vector $\left(v(x), m(x), \lambda, \bar{\Omega}\right)$ such that $(v(x), m(x), \lambda)$ solves (4.22) in the neighborhood $\bar{\Omega}$. Then the optimal control $u(x)$ for the local optimum problem is still given by (4.23).

4.4.6 Problem 2: Dynamically Extended Problem

We define another mean field game problem that differs from Problem 1 because of the players' dynamics. In particular we consider a new state $z(t) : [0, \infty[\longmapsto \mathbb{R}^{2d}$, that will be referred to as extended state and that is defined as follows

$$z(t) = \left(x^T(t), \xi^T(t)\right)^T$$

where $x(t) : [0, \infty[\longmapsto \mathbb{R}^d$, $\xi(t) : [0, \infty[\longmapsto \mathbb{R}^d$ and $\xi(t)$ is referred to as **dynamic** extension. Moreover the dynamics of $z(t)$ is

$$\dot{z} = \begin{pmatrix} \dot{x} \\ \dot{\xi} \end{pmatrix} = \begin{pmatrix} f_1(x) \\ \vdots \\ f_d(x) \\ - \\ \tau_1(x,\xi) \\ \vdots \\ \tau_d(x,\xi) \end{pmatrix} + \begin{pmatrix} g_{1,1}(x) & \cdots & g_{1,m}(x) \\ \vdots & & \vdots \\ g_{d,1}(x) & \cdots & g_{d,m}(x) \\ - & - & - \\ & & \\ & 0 & \\ & & \end{pmatrix} \begin{pmatrix} u_1(x) \\ \vdots \\ u_m(x) \end{pmatrix} + \tag{4.38}$$

$$+ \begin{pmatrix} \sigma_{1,1} & \cdots & \sigma_{1,d} \\ \vdots & & \vdots \\ \sigma_{d,1} & \cdots & \sigma_{d,d} \\ - & - & - \\ & & \\ & 0 & \\ & & \end{pmatrix} \begin{pmatrix} \dot{W}_1(t) \\ \vdots \\ \dot{W}_d(t) \end{pmatrix}$$

where the dynamics of $x(t)$ is described in (4.12) and

$$\dot{\xi} = \begin{pmatrix} \tau_1(x,\xi) & \dots & \tau_d(x,\xi) \end{pmatrix}^T = \tau(x,\xi) : \mathbb{R}^{2d} \longmapsto \mathbb{R}^d$$

is the dynamics of $\xi(t)$ that can be freely chosen. Moreover the initial condition of $\xi(t)$ is $\xi(0) = \xi_0$. The dynamics (4.38) can be more compactly rewritten as

$$\dot{z} = f_{ext}(x,\xi) + g_{ext}(x)u(x) + \sigma_{ext}\dot{W}(t) \quad z(0) = z_0$$

where

- $z = \begin{pmatrix} x \\ \xi \end{pmatrix} \in \mathbb{R}^{2d}$ and consequently $\dot{z} = \begin{pmatrix} \dot{x} \\ \dot{\xi} \end{pmatrix} \in \mathbb{R}^{2d}$
- $f_{ext}(x) = \begin{pmatrix} f(x) \\ -- \\ \tau(x,\xi) \end{pmatrix} \in \mathbb{R}^{2d}$
- $g_{ext}(x) = \begin{pmatrix} g(x) \\ -- \\ 0 \end{pmatrix} \in \mathbb{R}^{2d \times b}$

- $\sigma_{ext} = \begin{pmatrix} \sigma \\ -- \\ 0 \end{pmatrix} \in \mathbb{R}^{2d \times d}$

- $z_0 = \begin{pmatrix} x_0 \\ \xi_0 \end{pmatrix} \in \mathbb{R}^{2d}$

On the contrary the cost function is not linked to the dynamic extension therefore it is still (4.14). The population density function is consequently defined as $m_{ext}(x, \xi) : \mathbb{R}^{2d} \longmapsto \mathbb{R}$ and the neighborhood of the origin where we want to solve the problem is $\bar{\Omega}_{ext} \subset \mathbb{R}^{2d}$.

This problem will be referred to as dynamically extended problem or Problem 2 and it is fully described by the following system

$$\begin{cases} \dot{z} = f_{ext}(x, \xi) + g_{ext}(x)u(x) + \sigma_{ext}\dot{W}(t) \quad z(0) = z_0 \\ J(x_0, \xi_0, u(x, \xi)) := \lim_{T\to\infty} \frac{1}{T} E\left[\int_0^T \left(\frac{(u(x,\xi))^T E\, u(x,\xi)}{2} + V[m](x, \xi)\right) dt\right] \\ u(x, \xi) = \arg\min_{u(x,\xi)\in U_{\bar{\Omega}_{ext}}} (J(x_0, u(x))) \\ \int_{\mathbb{R}^{2d}} m_{ext}(x, \xi) dx\, d\xi = 1 \end{cases} \tag{4.39}$$

The solution $(u(x, \xi), m(x, \xi), \tau(x, \xi))$ of (4.39) will be referred to as **dynamic local** solution of (4.22).

In order to find the solution of the dynamically extended problem (4.39) we consider the corresponding HJB FPK PDE system. We also remark the fact that the extended value function $v_{ext}(x, \xi) : \mathbb{R}^{2d} \longmapsto \mathbb{R}$ is now defined as

$$v_{ext}(x, \xi) = \inf_{u(x,\xi)\in U_{\bar{\Omega}_{ext}}} J(x_0, \xi_0, u(x, \xi))$$

Nevertheless, in order to simplify the notation, we will use $v_{ext}(x, \xi) = v(x, \xi)$, $m_{ext}(x, \xi) = m(x, \xi)$ and $\bar{\Omega}_{ext} = \bar{\Omega}$. The derivatives of $v(x, \xi)$ and $m(x, \xi)$ are

$$v_z(z) = \frac{\partial v}{\partial z} = \left(\frac{\partial v}{\partial x_1} \cdots \frac{\partial v}{\partial x_d}, \frac{\partial v}{\partial \xi_1} \cdots \frac{\partial v}{\partial \xi_d}\right) = (v_x(x, \xi), v_\xi(x, \xi)) \in \mathbb{R}^{2d}$$

$$m_z(z) = \frac{\partial m}{\partial z} = \left(\frac{\partial m}{\partial x_1} \cdots \frac{\partial m}{\partial x_d}, \frac{\partial m}{\partial \xi_1} \cdots \frac{\partial m}{\partial \xi_d}\right) = (m_x(x, \xi), m_\xi(x, \xi)) \in \mathbb{R}^{2d}$$

$$v_{zz}(z) = \frac{\partial^2 v}{\partial z^2} = \begin{pmatrix} \frac{\partial^2 v}{\partial x_1^2} & \cdots & \frac{\partial^2 v}{\partial x_1 x_d} & | & \frac{\partial^2 v}{\partial x_1 \xi_1} & \cdots & \frac{\partial^2 v}{\partial x_1 \xi_d} \\ \vdots & \ddots & \vdots & | & \vdots & \ddots & \vdots \\ \frac{\partial^2 v}{\partial x_d x_1} & \cdots & \frac{\partial^2 v}{\partial x_d^2} & | & \frac{\partial^2 v}{\partial x_d \xi_1} & \cdots & \frac{\partial^2 v}{\partial x_d \xi_d} \\ -- & -- & -- & + & -- & -- & -- \\ \frac{\partial^2 v}{\partial \xi_1 x_1} & \cdots & \frac{\partial^2 v}{\partial \xi_1 x_d} & | & \frac{\partial^2 v}{\partial \xi_1^2} & \cdots & \frac{\partial^2 v}{\partial \xi_1 \xi_d} \\ \vdots & \ddots & \vdots & | & \vdots & \ddots & \vdots \\ \frac{\partial^2 v}{\partial \xi_d x_1} & \cdots & \frac{\partial^2 v}{\partial \xi_d x_d} & | & \frac{\partial^2 v}{\partial \xi_d \xi_1} & \cdots & \frac{\partial^2 v}{\partial \xi_d^2} \end{pmatrix} = \begin{pmatrix} v_{xx} & | & v_{x\xi} \\ -- & + & -- \\ v_{\xi x} & | & v_{\xi\xi} \end{pmatrix}$$

$$m_{zz}(z) = \frac{\partial^2 m}{\partial z^2} = \begin{pmatrix} \frac{\partial^2 m}{\partial x_1^2} & \cdots & \frac{\partial^2 m}{\partial x_1 x_d} & | & \frac{\partial^2 m}{\partial x_1 \xi_1} & \cdots & \frac{\partial^2 m}{\partial x_1 \xi_d} \\ \vdots & \ddots & \vdots & | & \vdots & \ddots & \vdots \\ \frac{\partial^2 m}{\partial x_d x_1} & \cdots & \frac{\partial^2 m}{\partial x_d^2} & | & \frac{\partial^2 m}{\partial x_d \xi_1} & \cdots & \frac{\partial^2 m}{\partial x_d \xi_d} \\ -- & -- & -- & + & -- & -- & -- \\ \frac{\partial^2 m}{\partial \xi_1 x_1} & \cdots & \frac{\partial^2 m}{\partial \xi_1 x_d} & | & \frac{\partial^2 m}{\partial \xi_1^2} & \cdots & \frac{\partial^2 m}{\partial \xi_1 \xi_d} \\ \vdots & \ddots & \vdots & | & \vdots & \ddots & \vdots \\ \frac{\partial^2 m}{\partial \xi_d x_1} & \cdots & \frac{\partial^2 m}{\partial \xi_d x_d} & | & \frac{\partial^2 m}{\partial \xi_d \xi_1} & \cdots & \frac{\partial^2 m}{\partial \xi_d^2} \end{pmatrix} = \begin{pmatrix} m_{xx} & | & m_{x\xi} \\ -- & + & -- \\ m_{\xi x} & | & m_{\xi\xi} \end{pmatrix}$$

where $v_{zz}(z), m_{zz}(z) \in \mathbb{R}^{2d\times 2d}$. The HJB FPK PDE system for the extended problem, where the dependencies of state z are neglected, is the following

$$\begin{cases} -tr(\nu_{ext} v_{zz}) + v_z\, f_{ext} - \frac{1}{2} v_z g_{ext} E^{-1} g_{ext}^T v_z^T + \lambda = \\ -q(x) + tr\left(C(x)\ \left(Var(m)\right)^{-1}\right) + d \ln{(m)} \\ \\ -tr(\nu_{ext} m_{zz}) - div\left(+m\, f_{ext} - m\ v_z g_{ext} E^{-1} g_{ext}^T\right) = 0 \end{cases} \tag{4.40}$$

where

$$\nu_{ext} = \frac{\sigma_{ext}\sigma_{ext}^T}{2} = \frac{1}{2}\begin{pmatrix} \sigma \\ -- \\ 0 \end{pmatrix}\begin{pmatrix} \sigma \\ -- \\ 0 \end{pmatrix}^T = \frac{1}{2}\begin{pmatrix} \sigma\sigma^T & | & 0 \\ -- & | & -- \\ 0 & | & 0 \end{pmatrix} = \begin{pmatrix} \nu & | & 0 \\ -- & | & -- \\ 0 & | & 0 \end{pmatrix}$$

Note that

$$\begin{aligned} tr(\nu_{ext} v_{zz}) &= tr\left(\begin{pmatrix} \nu & | & 0 \\ -- & | & -- \\ 0 & | & 0 \end{pmatrix}\begin{pmatrix} v_{xx} & | & v_{x\xi} \\ -- & + & -- \\ v_{\xi x} & | & v_{\xi\xi} \end{pmatrix}\right) \\ &= tr\begin{pmatrix} \nu v_{xx} & | & \nu v_{x\xi} \\ -- & + & -- \\ 0 & | & 0 \end{pmatrix} = tr(\nu v_{xx}) \end{aligned}$$

$$v_z\, f_{ext} \;=\; \begin{pmatrix} v_x & | & v_\xi \end{pmatrix} \begin{pmatrix} f \\ -- \\ \dot{\xi} \end{pmatrix} = v_x f + v_\xi \dot{\xi}$$

$$\begin{aligned} v_z g_{ext} E^{-1} g_{ext}^T v_z^T &= \begin{pmatrix} v_x & | & v_\xi \end{pmatrix} \begin{pmatrix} g \\ -- \\ 0 \end{pmatrix} E^{-1} \begin{pmatrix} g \\ -- \\ 0 \end{pmatrix}^T \begin{pmatrix} v_x & | & v_\xi \end{pmatrix}^T \\ &= v_x g E^{-1} g^T v_x^T \end{aligned}$$

Finally, (4.40) can be rewritten as

$$\begin{cases} -tr(\nu v_{xx}) + v_x\, f(x) + v_\xi \dot{\xi} - \frac{1}{2} v_x g E^{-1} g^T v_x^T + \lambda = \\ -q(x) + d \ln(m) \\ \\ -tr(\nu m_{xx}) - div\left(+m \begin{pmatrix} f^T & | & \dot{\xi}^T \end{pmatrix} - m\, v_x g E^{-1} g^T\right) = 0 \end{cases} \tag{4.41}$$

In this extended mean field game a solution of the local optimum problem is expressed as $\left(\tau(x,\xi), m(x,\xi), v(x,\xi), \lambda, \bar{\Omega}\right)$ where $\bar{\Omega} \subseteq \mathbb{R}^{2d}$ is the neighborhood of the origin where $(\tau(x,\xi), m(x,\xi), v(x,\xi), \lambda)$ solves (4.40). The optimal control for the extended mean field game is given by

$$u_{opt}(x,\xi) = -E^{-1} g^T(x) v_x(x,\xi)^T \tag{4.42}$$

We note that in this case we have a sort of new degree of freedom because we can freely choose $\dot{\xi} = \tau(x,\xi)$ and we may use it to make it easier to solve (4.41). However (4.40) is still very difficult or impossible to solve in the general case.

4.4.7 Problem 3: Approximate Problem

In order to find an easier way to compute a solution for the Problem 2 without solving PDEs, we introduce a new class of problems. The problems in this class differ from Problem 2 because they have a different cost function and a different noise contribution. Firstly, we consider a dynamics similar to the one in (4.38) but with a smaller noise contribution namely

$$\begin{pmatrix} \dot{x} \\ \dot{\xi} \end{pmatrix} = \begin{pmatrix} f_1(x) \\ \vdots \\ f_d(x) \\ - \\ \tau_1(x,\xi) \\ \vdots \\ \tau_d(x,\xi) \end{pmatrix} + \begin{pmatrix} g_{1,1}(x) & \cdots & g_{1,m}(x) \\ \vdots & & \vdots \\ g_{d,1}(x) & \cdots & g_{d,m}(x) \\ - & - & - \\ & 0 & \end{pmatrix} \begin{pmatrix} u_1(t,x) \\ \vdots \\ u_m(t,x) \end{pmatrix} + \frac{1}{\alpha} \begin{pmatrix} \sigma_{1,1} & \cdots & \sigma_{1,d} \\ \vdots & & \vdots \\ \sigma_{d,1} & \cdots & \sigma_{d,d} \\ - & - & - \\ & 0 & \end{pmatrix} \begin{pmatrix} W_1(t) \\ \vdots \\ W_d(t) \end{pmatrix} \tag{4.43}$$

where $\alpha \geq 1$ is a constant. Obviously, if $\alpha = 1$, (4.43) is equal to the dynamics of Problem 2.

Now we consider the following cost function

$$J(x_0, \xi_0, u) := \lim_{T \to \infty} \frac{1}{T} E \left[\int_0^T \left(\frac{u(x,\xi)^T E(x)\, u(x,\xi)}{2} + V[m](x,\xi) + \Upsilon \right) dt \right] \tag{4.44}$$

where $\Upsilon \geq 0$ is a constant. If $\Upsilon = 0$ (4.44) is equal to cost function of the Problem 2.

Each problem of the previously described class is identified by the pair of constants (α, Υ). It will be referred to as approximate problem or Problem 3 and it is fully described by the following system

$$\begin{cases} \dot{z} = f_{ext}(x,\xi) + g_{ext}(x)u(x) + \frac{1}{\alpha}\sigma_{ext}\dot{W}(t) \quad z(0) = z_0 \\ J\left(x_0, \xi_0, u(x,\xi)\right) := \lim_{T\to\infty} \frac{1}{T} \, E\left[\int_0^T \left(\frac{(u(x,\xi))^T E \, u(x,\xi)}{2} + V\left[m\right](x,\xi) + \Upsilon\right) dt\right] \\ u(x,\xi) = \arg\min_{u(x,\xi)\in U_{\bar{\Omega}_{ext}}} \left(J\left(x_0, u(x)\right)\right) \\ \int_{\mathbb{R}^{2d}} m_{ext}(x,\xi) dx \, d\xi = 1 \end{cases} \tag{4.45}$$

The solution $(u(x,\xi), m(x,\xi), \tau(x,\xi))$ of one problem of the class (4.39) will be referred to as **local dynamic approximate** solution of (4.22).

The system of an HJB and a FPK equation corresponding to the class of problems (4.45) is given by

$$\begin{cases} -tr\left(\nu_{new} v_{xx}\right) + v_x \, f + v_\xi \dot{\xi} - \frac{1}{2} v_x g E^{-1} g^T v_x^T + \lambda = \\ -\Upsilon - q + d\ln\left(m\right) \\ \\ -tr(\nu_{new} m_{xx}) - div\left(+m \left(\begin{array}{ccc} f^T & | & \dot{\xi}^T \end{array}\right) - m \; v_x g E^{-1} g^T\right) = 0 \end{cases} \tag{4.46}$$

where $\nu_{new} \in \mathbb{R}^{d\times d}$ is computed as follows

$$\begin{aligned} \left(\begin{array}{ccc} \nu_{new} & | & 0 \\ -- & + & -- \\ 0 & | & 0 \end{array}\right) &= \frac{1}{2}\left(\left(\begin{array}{c} \sigma \\ - \\ 0 \end{array}\right) - \frac{\alpha-1}{\alpha}\left(\begin{array}{c} \sigma \\ - \\ 0 \end{array}\right)\right)\left(\left(\begin{array}{c} \sigma \\ - \\ 0 \end{array}\right) - \frac{\alpha-1}{\alpha}\left(\begin{array}{c} \sigma \\ - \\ 0 \end{array}\right)\right)^T \\ &= \frac{1}{\alpha^2}\left(\begin{array}{ccc} \frac{1}{2}\sigma\sigma^T & | & 0 \\ -- & + & -- \\ 0 & | & 0 \end{array}\right) \\ &= \left(\begin{array}{ccc} \frac{1}{2}\sigma\sigma^T & | & 0 \\ -- & + & -- \\ 0 & | & 0 \end{array}\right) - \frac{\alpha^2-1}{\alpha^2}\left(\begin{array}{ccc} \frac{1}{2}\sigma\sigma^T & | & 0 \\ -- & + & -- \\ 0 & | & 0 \end{array}\right) \end{aligned}$$

Therefore

$$\begin{aligned} \nu_{new} &= \left(1 - \frac{\alpha^2-1}{\alpha^2}\right)\left(\frac{1}{2}\sigma\sigma^T\right) \\ &= \nu - \frac{\alpha^2-1}{\alpha^2}\nu = \nu - \nu_{gap}(\alpha) \end{aligned} \tag{4.47}$$

where

$$\nu_{gap}(\alpha) = \frac{\alpha^2 - 1}{\alpha^2}\nu$$

hence $\nu_{gap}(\alpha) \in \mathbb{R}^{d\times d}$ is a positive definite matrix because ν is positive definite.

Substituting (4.47) in (4.46) we have

$$\begin{cases} -tr\left(\left(\nu - \nu_{gap}(\alpha)\right) v_{xx}\right) + v_x\, f + v_\xi \dot{\xi} - \frac{1}{2} v_x g E^{-1} g^T v_x^T + \lambda = \\ -\Upsilon - q + d\ln\left(m\right) \\ \\ -tr\left(\left(\nu - \nu_{gap}(\alpha)\right) m_{xx}\right) - div\left(+m \begin{pmatrix} f^T & | & \dot{\xi}^T \end{pmatrix} - m\, v_x g E^{-1} g^T\right) = 0 \end{cases}$$

Separating the terms depending on Υ and α we obtain

$$\begin{cases} -tr\left(\nu v_{xx}\right) + v_x\, f + v_\xi \dot{\xi} - \frac{1}{2} v_x g E^{-1} g^T v_x^T + \lambda + q - d\ln\left(m\right) \\ = -\Upsilon - tr\left(\nu_{gap}(\alpha) v_{xx}\right) \\ \\ -tr\left(\nu m_{xx}\right) - div\left(+m \begin{pmatrix} f^T & | & \dot{\xi}^T \end{pmatrix} - m\, v_x g E^{-1} g^T\right) = -tr\left(\nu_{gap}(\alpha) m_{xx}\right) \end{cases} \tag{4.48}$$

In order to make (4.48) easier to solve we need to introduce the following two assumptions.

Assumption 1 $tr\left(\nu m_{xx}(0,0)\right) \leq 0$. This will be a steady assumption from now on.

Remark 4.13. If Assumption 1 holds, $tr\left(\nu_{gap}(\alpha) m_{xx}(0,0)\right) \leq 0$ because

$$tr\left(\nu_{gap}(\alpha) m_{xx}\right) = \frac{\alpha^2 - 1}{\alpha^2} tr\left(\nu m_{xx}\right) \leq 0$$

Remark 4.14. Note that Assumption 1 holds if we have a bell shaped population density function. Indeed, for example, Assumption 1 is verified if the population density function is concave at $(x, \xi) = (0, 0)$ because of the properties of trace operator (A.6) as explained in Appendix A.

Assumption 2 There exists a neighborhood Ω_{ass} of the origin such that

$$0 \leq -tr\left(\nu m_{xx}\right) - div\left(+m \begin{pmatrix} f^T & | & \dot{\xi}^T \end{pmatrix} - m\, v_x g E^{-1} g^T\right) < -tr\left(\nu m_{xx}\right) \tag{4.49}$$

for each $(x, \xi) \in \Omega_{ass} \subseteq \mathbb{R}^{2d}$. This will be a steady assumption from now on. If Assumption 1 holds, since all functions are smooth, such neighborhood Ω_{ass} always exists.

Remark 4.15. A necessary condition for Assumption 2 to hold is that $tr\,(\nu m_{xx}) \leq 0$ for all $(x, \xi) \in \Omega_{ass}$. For example, in the one-dimensional case, it means that we can find only population density functions that are concave in a neighborhood of the origin.

If Assumptions 1 and 2 hold then solving a problem of the class (4.45) and consequently one of the systems of PDEs (4.48) in a neighborhood of the origin $\bar{\Omega} \subseteq \Omega_{ass}$ is equivalent to finding a vector $\left(\tau(x,\xi), m(x,\xi), v(x,\xi), \lambda, \bar{\Omega}\right)$ such that the following hold

$$\begin{cases} -tr\,(\nu v_{xx}) + v_x\, f + v_\xi \dot{\xi} - \frac{1}{2} v_x g E^{-1} g^T v_x^T + \lambda + q - d \ln{(m)} \leq 0 \\ \\ -tr\,(\nu m_{xx}) - div\left(+m \begin{pmatrix} f^T & | & \dot{\xi}^T \end{pmatrix} - m\, v_x g E^{-1} g^T\right) \geq 0 \end{cases} \tag{4.50}$$

Indeed, if the solution $\left(\tau(x,\xi), m(x,\xi), v(x,\xi), \lambda, \bar{\Omega}\right)$ found from (4.50) is such that

$$\begin{cases} -tr\,(\nu v_{xx}) + v_x\, f + v_\xi \dot{\xi} - \frac{1}{2} v_x g E^{-1} g^T v_x^T + \lambda + q - d \ln{(m)} = \Delta_1 \\ \\ -tr\,(\nu m_{xx}) - div\left(+m \begin{pmatrix} f^T & | & \dot{\xi}^T \end{pmatrix} - m\, v_x g E^{-1} g^T\right) = \Delta_2 \end{cases}$$

where $\Delta_1 \leq 0$ and $0 \leq \Delta_2 < -tr\,(\nu m_{xx})$ because of Assumption 2, then we have that $\left(\tau(x,\xi), m(x,\xi), v(x,\xi), \lambda, \bar{\Omega}\right)$ solves the problem of the class (4.48) corresponding to

$$a = \sqrt{\frac{tr\,(\nu m_{xx})}{\Delta_2 + tr\,(\nu m_{xx})}} \qquad \Upsilon = -\Delta_1 + \Delta_2 \frac{tr\,(\nu v_{xx})}{tr\,(\nu m_{xx})}$$

Moreover the solution of the corresponding approximate problem is $(\tau(x,\xi), m(x,\xi), u(x,\xi))$ where $u(x,\xi)$ is given by (4.42).

4.4.8 Procedure Summary

To sum up, the method that we propose to solve the stationary mean field game (4.15) and to make each player reach the equilibrium consists of

1. Computing the **local dynamic approximate** solution $(\hat{\tau}(x,\xi), \hat{m}(x,\xi), \hat{u}(x,\xi))$ of (4.15)

2. Constructing the **extended dynamics** (4.38) of the mean field game (4.15) with $\dot{\xi} = \hat{\tau}(x,\xi)$

3. Using $\hat{u}(x,\xi)$ as a control function for each player of the mean field game (4.39)

If the distribution of players' state has the same form as the probability density function $\hat{m}(x,\xi)$ and if (4.16) holds we can say that the solution is good. Further analysis would be necessary to understand the accuracy of the solution $(\tau(x,\xi), m(x,\xi), u(x,\xi))$ without performing simulations. We simply note that, obviously, a local dynamic approximate solution is preferable to another one if the corresponding validity neighborhood $\bar{\Omega}$ is larger. A numerical example that shows the effectiveness of the local dynamic approximate solution will be provided in Section 4.5.

According to what we have shown, in order to compute a dynamic approximate local solution of (4.15) it is sufficient to solve (4.50). Nevertheless, (4.50) is a system of partial differential inequalities (PDIs) therefore it may still be difficult to be solved. For this reason, in the following chapter, we propose a way to solve it without dealing with PDIs.

4.4.9 Algebraic Mean Field Game Solution

In order to find an equilibrium for (4.48) solving only a system of algebraic equations, we introduce the following definition.

Definition 4.16. (Algebraic Mean Field Game solution) An *algebraic mean field game solution (algebraic MFGS)* in $\Omega_{alg}$of the problem (4.15) is defined as the pair $(P(\cdot), G(\cdot))$ such that

- $P(\cdot) : \mathbb{R}^d \longmapsto \mathbb{R}^{d\times d}$, $G(\cdot) : \mathbb{R}^d \longmapsto \mathbb{R}^{d\times d}$
- $G(x)$ is such that $\int_{\mathbb{R}^d} \exp\left(-\frac{1}{2}x^T G(x) x\right) dx < \infty$
- $P(\cdot)$ and $G(\cdot)$ are symmetric i.e. $P(\cdot) = P(\cdot)^T$ and $G(\cdot) = G(\cdot)^T$

- $P(\cdot)$ and $G(\cdot)$ are C^2 mappings
- the following holds for all $x \in \Omega_{alg}$, where Ω_{alg} is a neighborhood of the origin, and for some mappings $\Sigma_1, \Sigma_2 : \mathbb{R}^d \longmapsto \mathbb{R}^{d\times d}$ such that $\Sigma_2(x) \geq 0$ and $\Sigma_1(x) \geq 0$ for all $x \in \Omega_{alg}$

$$\begin{cases} -x^T P(x) g E^{-1} g^T G(x) x + \frac{1}{2} x^T F(x)^T G(x) x + \\ +\frac{1}{2} x^T G(x) F(x) x - x^T G(x) \nu G(x) x + \\ +tr\left(P(x) g E^{-1} g^T\right) + \left(x^T P(x)\right) \bar{\nabla}_x \left(g(x) E^{-1} g(x)^T\right) + \\ -tr\left(+F(x)^T\right) - x^T \bar{\nabla}_x \left(F(x)\right) + tr\left(G(x)\nu\right) + \\ = +x^T \Sigma_2(x) x \\ \\ -\frac{1}{2} P(x) g E^{-1} g^T P(x) + \frac{1}{2} F(x)^T P(x) \\ +\frac{1}{2} P(x) F(x) + Q + \frac{1}{2} d G(x) \\ +\Sigma_1(x) = 0 \end{cases} \tag{4.51}$$

Remark 4.17. Given a problem (4.15), a mean field game solution may not exist because (4.51) may not be solvable.

Remark 4.18. Both $P(x)$ and $G(x)$ are symmetric matrices. Therefore, in order to compute their value, we need to find $2\left(\frac{d^2-d}{2} + d\right) = d^2 + d$ functions depending on x. However, the first equation of (4.51) is one-dimensional and the second one is d-dimensional therefore we have only $d+1$ relations. This fact clearly shows that we do not find only one particular solution but a set of solutions for the Problem 3. The additional degrees of freedom may be used to find a pair $(P(\cdot), G(\cdot))$ that makes the set Ω_{alg} as large as possible.

4.4.10 Proposed solution

Exploiting the previously defined concept of algebraic MFGS we want to provide a vector $\left(\tau(x,\xi), m(x,\xi), v(x,\xi), \lambda, \bar{\Omega}\right)$ that allows to solve the system of PDIs (4.48).

Theorem 4.19. *Given an algebraic MFGS $(P(\cdot), G(\cdot))$ of the problem (4.15) in Ω_{alg}, then a solution $\left(\tau(x,\xi), m(x,\xi), v(x,\xi), \lambda, \bar{\Omega}\right)$ of the system of PDIs (4.48) is the following*

$$v(x,\xi) = \frac{1}{2} x^T P(\xi) x + \frac{1}{2} (x-\xi)^T R (x-\xi) \tag{4.52}$$

$$m(x,\xi) = \kappa \exp\left(-\frac{1}{2}x^T G(\xi) x - \frac{1}{2}(x-\xi)^T S (x-\xi)\right) \tag{4.53}$$

$$\dot{\xi} = \tau(x,\xi) = -k\left(v_\xi(x,\xi)^T + \frac{m_\xi(x,\xi)^T}{m(x,\xi)}\right) \tag{4.54}$$

$$\lambda \;\; = - \;\; \max_{\Omega_2}\left(-tr\left(P(x)\nu\right) - d\ln\left(\kappa\right)\right) \tag{4.55}$$

$$\bar{\Omega} = \Omega_1 \cap \Omega_2 \cap \Omega_{alg,ext} \tag{4.56}$$

where

- $R,\ S \in \mathbb{R}^{d\times d}$ *are symmetric positive definite matrices which can be arbitrarily chosen in such a way that*

  $$tr\left(R\right) > tr\left(S\right)$$

- $\kappa \in \mathbb{R}$ *is the constant such that*

  $$\int_{\mathbb{R}^{2d}} \kappa \exp\left(-\frac{1}{2}x^T G(\xi) x - \frac{1}{2}(x-\xi)^T S (x-\xi)\right) dx\, d\xi = 1$$

  that always exists because of the assumptions on $G(\cdot)$

- $k > 0$ *is a constant that can be arbitrarily chosen*

- Ω_2 *is a neighborhood of the origin where the following inequality holds*

  $$-tr\left(\nu m_{xx}\right) - div\left(+m\begin{pmatrix} f^T & | & \dot{\xi}^T \end{pmatrix} - m\ v_x g E^{-1} g^T\right) \geq 0$$

- Ω_1 *is a neighborhood of the origin where the following inequality holds*

  $$-tr(\nu v_{xx}) + v_x\ f(x) + v_\xi \dot{\xi} - \frac{1}{2} v_x g(x) E^{-1} g^T(x) v_x^T + V[m](x) + \lambda \leq 0$$

- $\Omega_{alg,ext}$ *is the extension into space* $\mathbb{R}^{2d}$ *of* Ω_{alg} *namely*

  $$\Omega_{alg,ext} = \Omega_{alg} \times \mathbb{R}^d$$

Proof. A complete and detailed proof of the theorem that is one of the main results of this work is provided in the following. The proof consist on the substitution of the proposed solution $\left(\tau(x,\xi), m(x,\xi), v(x,\xi), \lambda, \bar{\Omega}\right)$ in the PDIs (4.50) and on their evaluation on the neighborhood of the origin $\bar{\Omega}$. Moreover it exploits some properties of the divergence operator and of trace operator.

Consider the algebraic MFGS $(P(\cdot), G(\cdot))$. From $P(\xi)$ and $G(\xi)$, we can also automatically define the functions $\Phi(x,\xi)$, $\bar{\Phi}(x,\xi)$, $\Lambda(x,\xi)$ and $\bar{\Lambda}(x,\xi)$ that take the following expressions

$$x^T\left(P(x) - P(\xi)\right) = (x-\xi)^T\,\Phi(x,\xi)^T$$

$$P(x) - P(\xi) = \bar{\Phi}(x,\xi)^T$$

$$x^T\left(G(x) - G(\xi)\right) = (x-\xi)^T\,\Lambda(x,\xi)^T$$

$$G(x) - G(\xi) = \bar{\Lambda}(x,\xi)^T$$

The derivatives of (4.52) and (4.53) are computed as follows

$$v_x = x^T P(\xi) + (x-\xi)^T R = x^T P(x) + (x-\xi)^T\left(R - \Phi(x,\xi)\right)^T$$

$$v_{xx} = P(\xi) + R = P(x) + R - \bar{\Phi}(x,\xi)^T$$

$$v_\xi = x^T \frac{\partial\left(\frac{1}{2}P(\xi)x\right)}{\partial\xi} - (x-\xi)^T R = x^T\Psi(x,\xi) - (x-\xi)^T R$$

$$m_x = -m(x,\xi)\left(x^T G(\xi) + (x-\xi)^T S\right) = -m(x,\xi)\left(x^T G(x) + (x-\xi)^T\left(S - \Lambda(x,\xi)\right)^T\right)$$

$$\begin{aligned} m_{xx} &= -m(x,\xi)\left(G(\xi)+S\right) - m(x,\xi)\left(x^T G(\xi) + (x-\xi)^T S\right)\left(x^T G(\xi) + (x-\xi)^T S\right)^T \\ &= -m(x,\xi)\left(G(x) + S - \bar{\Lambda}(x,\xi)^T\right) + \\ &\quad -m(x,\xi)\left(x^T G(x) + (x-\xi)^T\left(S-\Lambda(x,\xi)\right)^T\right)\left(x^T G(x) + (x-\xi)^T\left(S-\Lambda(x,\xi)\right)^T\right)^T \end{aligned}$$

$$m_\xi = -m(x,\xi)\left(x^T\frac{\partial\left(\frac{1}{2}G(\xi)x\right)}{\partial\xi} - (x-\xi)^T S\right) = -m(x,\xi)\left(x^T\Xi(x,\xi) - (x-\xi)^T S\right)$$

where $\Psi(x,\xi) = \frac{\partial\left(\frac{1}{2}P(\xi)x\right)}{\partial\xi}$ and $\Xi(x,\xi) = \frac{\partial\left(\frac{1}{2}G(\xi)x\right)}{\partial\xi}$.

Some helpful divergence operator properties, that will be used in the proof, are recall in Appendix A.

FPK equation

In order to make the proof clearer, dependencies on x will be neglected from now on. The second equation of (4.50) reads

$$-tr\left(\nu m_{xx}\right) - div\left(+m\left(\begin{array}{ccc} f^T & | & \dot{\xi}^T \end{array}\right) - m\, v_x g E^{-1} g^T\right) \geq 0$$

Using the definition of divergence, we obtain

$$\begin{aligned} div\left(m\left(\begin{array}{ccc} f^T & | & \dot{\xi}^T \end{array}\right)\right) &= \frac{\partial m f_1}{\partial x_1} + \cdots + \frac{\partial m f_d}{\partial x_d} + \frac{\partial m \dot{\xi}_1}{\partial \xi_1} + \cdots + \frac{\partial m \dot{\xi}_d}{\partial \xi_d} \\ &= div_x\left(m f^T\right) + div_\xi\left(m\, \dot{\xi}^T\right) \end{aligned}$$

where $div_x\left(m f^T\right) = \frac{\partial m f_1}{\partial x_1} + \cdots + \frac{\partial m f_d}{\partial x_d}$ and $div_\xi\left(m\, \dot{\xi}^T\right) = \frac{\partial m \dot{\xi}_1}{\partial \xi_1} + \cdots + \frac{\partial m \dot{\xi}_d}{\partial \xi_d}$. Consequently (4.50) can be written as

$$-tr(\nu m_{xx}) - div_x\left(+m\, f^T - m\, v_x g E^{-1} g^T\right) - div_\xi\left(+m\, \dot{\xi}^T\right) \geq 0$$

Exploiting (A.10), we have

$$-div_x\left(m_x \nu + m\, f^T - m\, v_x g E^{-1} g^T\right) - div_\xi\left(m\, \dot{\xi}^T\right) \geq 0$$

Then, substituting (4.53) in the previous expression, we obtain

$$div_x\left[m\left(\left(x^T G(x) + (x-\xi)^T\left(S - \Lambda(x,\xi)\right)^T\right)\nu + v_x g E^{-1} g^T - f^T\right)\right] +$$

$$div_\xi\left(-m\, \dot{\xi}^T\right) \geq 0$$

Finally, substituting (4.52) in the previous expression, the above expression becomes

$$div_x\left[m\left(\left(x^T P(x) + (x-\xi)^T\left(R - \Phi(x,\xi)\right)^T\right) g E^{-1} g^T - f^T\right)\right] +$$

$$+div_x\left[m\left(x^T G(x) + (x-\xi)^T\left(S - \Lambda(x,\xi)\right)^T\right)\nu\right] + div_\xi\left(-m\, \dot{\xi}^T\right) \geq 0$$

Now (A.8) can be used in order to obtain the following expression

$$div_x\left[\left(\left(x^T P(x) + (x-\xi)^T\left(R - \Phi(x,\xi)\right)^T\right) g E^{-1} g^T - f^T\right)\right] m +$$

$$+div_x\left[\left(x^T G(x) + (x-\xi)^T\left(S - \Lambda(x,\xi)\right)^T\right)\nu\right] m + div_\xi\left(-\dot{\xi}^T\right) m +$$

$$+\left(\left(x^T P(x) + (x-\xi)^T\left(R - \Phi(x,\xi)\right)^T\right) g E^{-1} g^T - f^T\right) m_x^T +$$

$$+\left(x^T G(x) + (x-\xi)^T\left(S - \Lambda(x,\xi)\right)^T\right)\nu\, m_x^T - \dot{\xi}^T m_\xi^T \geq 0$$

In order to simplify the above expression, we define the variable $T(x,\xi,k)$ as

$$\begin{aligned} T(x,\xi,k) \;\; := \;\; & div_x\left[\left(\left(x^T P(x) + (x-\xi)^T\left(R-\Phi(x,\xi)\right)^T\right) gE^{-1}g^T - f^T\right)\right] \\ & + div_x\left[\left(x^T G(x) + (x-\xi)^T\left(S-\Lambda(x,\xi)\right)^T\right)\nu\right] + div_\xi\left(-\dot{\xi}^T\right) \end{aligned}$$

Then, using (A.10) again and remembering that $f(x)=F(x)x$ and that

$$\begin{aligned} \dot{\xi} \;\; = \;\; & -k\left(v_\xi(x,\xi)^T + \frac{m_\xi(x,\xi)^T}{m(x,\xi)}\right) \\ = \;\; & -k\left(\Psi(x,\xi)^T x - R\left(x-\xi\right)\right) + k\left(\Xi(x,\xi)^T x - S\left(x-\xi\right)\right) \end{aligned} \qquad (4.57)$$

we can write

$$\begin{aligned} T(x,\xi,k) \;\; = \;\; & tr\left(\left(P(x) + R - \bar{\Phi}(x,\xi)^T\right) gE^{-1}g^T\right) + \\ & + \left(x^T P(x) + (x-\xi)^T\left(R-\Phi(x,\xi)\right)^T\right)\bar{\nabla}_x\left(g(x)E^{-1}g(x)^T\right) + \\ & -tr\left(+F(x)^T\right) - x^T\bar{\nabla}_x\left(F(x)\right) + tr\left(\left(G(x) + S - \bar{\Lambda}(x,\xi)^T\right)\nu\right) + \\ & +k\, tr\left(x^T\frac{\partial\Psi(x,\xi)}{\partial\xi} + R\right) - k\, tr\left(x^T\frac{\partial\Xi(x,\xi)}{\partial\xi} + S\right) \end{aligned}$$

The expression that we are studying becomes

$$\begin{aligned} & \left(\left(x^T P(x) + (x-\xi)^T\left(R-\Phi(x,\xi)\right)^T\right) gE^{-1}g^T - f^T\right) m_x^T + \\ & + \left(x^T G(x) + (x-\xi)^T\left(S-\Lambda(x,\xi)\right)^T\right)\nu\, m_x^T - \dot{\xi}^T m_\xi^T + m\, T(x,\xi,k) \geq 0 \end{aligned}$$

Therefore, substituting the expressions for m_x and m_ξ and recalling again that $f(x)=F(x)x$ and that the expression of $\dot{\xi}$ is given by (4.57), we obtain

$$\begin{aligned} & -\left(x^T P(x) + (x-\xi)^T\left(R-\Phi(x,\xi)\right)^T\right) gE^{-1}g^T + \\ & + x^T F(x)^T\left(G(x)x + \left(S-\Lambda(x,\xi)\right)(x-\xi)\right) m\left(G(x)x + \left(S-\Lambda(x,\xi)\right)(x-\xi)\right) m + \\ & -\left(x^T G(x) + (x-\xi)^T\left(S-\Lambda(x,\xi)\right)^T\right)\nu\left(G(x)x + \left(S-\Lambda(x,\xi)\right)(x-\xi)\right) m + \\ & -k\left(x^T\Psi(x,\xi) - (x-\xi)^T R\right)\left(\Xi(x,\xi)^T x - S\left(x-\xi\right)\right) m \\ & +k\left(x^T\Xi(x,\xi) - (x-\xi)^T S\right)\left(\Xi(x,\xi)^T x - S\left(x-\xi\right)\right) m + m\, T(x,\xi,k) \geq 0 \end{aligned}$$

Note that, since $m(x,\xi) > 0$ for all (x,ξ), we can divide the previous expression by m

$$
\begin{aligned}
&-x^T P(x) g E^{-1} g^T \left(S - \Lambda(x,\xi)\right)(x-\xi) - (x-\xi)^T \left(R - \Phi(x,\xi)\right)^T g E^{-1} g^T G(x) x + \\
&-x^T P(x) g E^{-1} g^T G(x) x - (x-\xi)^T \left(R - \Phi(x,\xi)\right)^T g E^{-1} g^T \left(S - \Lambda(x,\xi)\right)(x-\xi) + \\
&+x^T F(x)^T \left(G(x)x + \left(S - \Lambda(x,\xi)\right)(x-\xi)\right) + \\
&-x^T G(x) \nu G(x) x - (x-\xi)^T \left(S - \Lambda(x,\xi)\right)^T \nu \left(S - \Lambda(x,\xi)\right)(x-\xi) + \\
&-x^T G(x) \nu \left(S - \Lambda(x,\xi)\right)(x-\xi) - (x-\xi)^T \left(S - \Lambda(x,\xi)\right)^T \nu G(x) x + \\
&-k x^T \Psi(x,\xi) \Xi(x,\xi)^T x - k (x-\xi)^T R S (x-\xi) + \\
&+k x^T \Psi(x,\xi) S (x-\xi) + k (x-\xi)^T R \Xi(x,\xi)^T x \\
&+k x^T \Xi(x,\xi) \Xi(x,\xi)^T x + k (x-\xi)^T S S (x-\xi) + \\
&-k x^T \Xi(x,\xi) S (x-\xi) - k (x-\xi)^T S \Xi(x,\xi)^T x + T(x,\xi,k) \geq 0
\end{aligned}
\tag{4.58}
$$

As $x^T V_1 V_2 x$ is scalar, the following expressions hold

$$x^T V_2^T V_1^T x = x^T V_1 V_2 x$$

$$x^T V_1 V_2 x = \frac{1}{2} x^T V_1 V_2 x + \frac{1}{2} x^T V_2^T V_1^T x \tag{4.59}$$

Exploiting (4.59), the inequality (4.58) can be written in the following form

$$
\begin{pmatrix} x \\ x-\xi \end{pmatrix}^T \left[\begin{pmatrix} M_{1,1}^{II} & M_{1,2}^{II} \\ M_{2,1}^{II} & M_{2,2}^{II} \end{pmatrix} + k \begin{pmatrix} N_{1,1}^{II} & N_{1,2}^{II} \\ N_{2,1}^{II} & N_{2,2}^{II} \end{pmatrix} - k \begin{pmatrix} D_{1,1} & D_{1,2} \\ D_{2,1} & D_{2,2} \end{pmatrix} \right] \begin{pmatrix} x \\ x-\xi \end{pmatrix} +
$$

$$+T(x,\xi,k) \geq 0$$

where

$$
\begin{aligned}
M^{II}_{1,1} &= -P(x)gE^{-1}g^TG(x) + \frac{1}{2}F(x)^TG(x) + \\
&\quad +\frac{1}{2}G(x)F(x) - G(x)\nu G(x) \\
M^{II}_{1,2} &= -\frac{1}{2}P(x)gE^{-1}g^T\left(S - \Lambda(x,\xi)\right) - G(x)\nu\left(S - \Lambda(x,\xi)\right) \\
&\quad -\frac{1}{2}G(x)gE^{-1}g^T\left(R - \Phi(x,\xi)\right) + \frac{1}{2}F(x)^T\left(S - \Lambda(x,\xi)\right) \\
M^{II}_{2,1} &= \left(M^{II}_{1,2}\right)^T \\
M^{II}_{2,2} &= -\left(R - \Phi(x,\xi)\right)^T gE^{-1}g^T\left(S - \Lambda(x,\xi)\right) + \\
&\quad -\left(S - \Lambda(x,\xi)\right)^T\nu\left(S - \Lambda(x,\xi)\right) \\
D^{II}_{1,1} &= +\Psi(x,\xi)\mathit{\Xi}(x,\xi)^T \\
D^{II}_{1,2} &= -\frac{1}{2}\Psi(x,\xi)S - \frac{1}{2}\mathit{\Xi}(x,\xi)R \\
D^{II}_{2,1} &= \left(D^{II}_{1,2}\right)^T \\
D^{II}_{2,2} &= +RS \\
N^{II}_{1,1} &= +\mathit{\Xi}((x,\xi)\mathit{\Xi}(x,\xi)^T \\
N^{II}_{1,2} &= -\mathit{\Xi}(x,\xi)S \\
N^{II}_{2,1} &= \left(N^{II}_{1,2}\right)^T \\
N^{II}_{2,2} &= SS
\end{aligned}
$$

Finally, remembering the Definition 4.16 of algebraic solution we obtain

$$\begin{pmatrix} x \\ x-\xi \end{pmatrix}^T \left[\begin{pmatrix} \Sigma_2(x) & M_{1,2}^{II} \\ M_{2,1}^{II} & M_{2,2}^{II} \end{pmatrix} + k \begin{pmatrix} N_{1,1}^{II} & N_{1,2}^{II} \\ N_{2,1}^{II} & N_{2,2}^{II} \end{pmatrix} - k \begin{pmatrix} D_{1,1} & D_{1,2} \\ D_{2,1} & D_{2,2} \end{pmatrix} \right] \begin{pmatrix} x \\ x-\xi \end{pmatrix} +$$

$$+tr\left(\left(-\bar{\Phi}(x,\xi)^T + R\right) gE^{-1}g^T\right) + tr\left(\left((x-\xi)^T\left(R - \Phi(x,\xi)\right)^T\right)\bar{\nabla}_x\left(g(x)E^{-1}g(x)^T\right)\right) +$$

$$+tr\left(\left(-\bar{\Lambda}(x,\xi)^T + S\right)\nu\right) + k\, tr\left(x^T\frac{\partial\Psi(x,\xi)}{\partial\xi} + R\right) - k\, tr\left(x^T\frac{\partial\Xi(x,\xi)}{\partial\xi} + S\right) \geq 0 \tag{4.60}$$

Evaluate (4.60) at $(x,\xi) = (0,0)$ and note that, by the way they have been defined, $\bar{\Phi}(x,\xi)^T = 0$ and $\bar{\Lambda}(x,\xi)^T = 0$. Hence (4.60) evaluated at $(x,\xi) = (0,0)$ reads

$$+tr\left(RgE^{-1}g^T\right) + tr\left(S\nu\right) + k\left(tr\left(R\right) - tr\left(S\right)\right) \geq 0$$

where$R$ and $gE^{-1}g^T$ are positive definite and hence $tr\left(RgE^{-1}g^T\right) > 0$. S and ν are both positive definite and therefore, due to (A.6) , $tr\left(S\nu\right) > 0$. Finally, for each $k \geq 0$, $k\left(tr\left(R\right) - tr\left(S\right)\right) \geq 0$ because $tr\left(R\right) > tr\left(S\right)$. It follows that (4.60) is strictly greater than zero at $(x,\xi) = (0,0)$. Moreover, noting that all functions in (4.60) are smooth, we deduce that a neighborhood of the origin where the inequality (4.60) holds exists. It is precisely $\Omega_2 \cap \Omega_{alg,ext}$.

HJB equation

The fist inequality of (4.50) is

$$-tr(\nu v_{xx}) + v_x\, f(x) + v_\xi\dot{\xi} - \frac{1}{2}v_x g(x)E^{-1}g^T(x)v_x^T + V[m](x) + \lambda \leq 0$$

where $V[m](x)$ reads

$$\begin{aligned} V[m](x) : &= x^TQ\,x + d\frac{1}{2}x^T\left(G(x) - \bar{\Lambda}(x,\xi)^T\right)x \\ & -d\ln\left(\kappa\right) + d\frac{1}{2}\left(x-\xi\right)^T S\left(x-\xi\right) \end{aligned}$$

Therefore, substituting (4.53) and (4.52) in the first inequality of (4.50), the first inequality of (4.50) can be written as

$$
\begin{aligned}
& tr\left(-\left(P(x)+R-\bar{\Phi}(x,\xi)^T\right)\nu\right)+ \\
& +\left(x^T P(x)+(x-\xi)^T\left(R-\Phi(x,\xi)\right)^T\right)F(x)x+ \\
& -\tfrac{1}{2}\left(x^T P(x)+(x-\xi)^T\left(R-\Phi(x,\xi)\right)^T\right)gE^{-1}g^T\left(x^T P(x)+(x-\xi)^T\left(R-\Phi(x,\xi)\right)^T\right)^T+ \\
& -k\left(x^T\Psi(x,\xi)-(x-\xi)^T R\right)\left(\Psi(x,\xi)^T x-R\left(x-\xi\right)\right)+ \\
& +k\left(x^T\Psi(x,\xi)-(x-\xi)^T R\right)\left(\Xi(x,\xi)^T x-S\left(x-\xi\right)\right)+ \\
& +d\tfrac{1}{2}x^T\left(G(x)-\bar{\Lambda}(x,\xi)^T\right)x+d\tfrac{1}{2}\left(x-\xi\right)^T S\left(x-\xi\right)+ \\
& -d\ln\left(\kappa\right)+x^T Q\,x+\lambda\leq 0
\end{aligned}
\tag{4.61}
$$

Finally (4.61) can be written in the following form

$$
\begin{aligned}
& \begin{pmatrix} x \\ x-\xi \end{pmatrix}^T\left[\begin{pmatrix} M^I_{1,1} & M^I_{1,2} \\ M^I_{2,1} & M^I_{2,2} \end{pmatrix}-k\begin{pmatrix} N^I_{1,1} & N^I_{1,2} \\ N^I_{2,1} & N^I_{2,2} \end{pmatrix}+k\begin{pmatrix} D_{1,1} & D_{1,2} \\ D_{2,1} & D_{2,2} \end{pmatrix}\right]\begin{pmatrix} x \\ x-\xi \end{pmatrix}+ \\
& +\lambda+tr\left(-\left(P(x)+R-\bar{\Phi}(x,\xi)^T\right)\nu\right)-d\ln\left(\kappa\right)\leq 0
\end{aligned}
$$

where

$$
\begin{aligned}
M^I_{1,1} &= -\frac{1}{2}P(x)gE^{-1}g^TP(x) + \frac{1}{2}F(x)^TP(x) + \\
&\quad + \frac{1}{2}P(x)F(x) + Q + \frac{1}{2}dG - \frac{1}{2}d\bar{\Lambda}(x,\xi)^T \\
M^I_{1,2} &= -\frac{1}{2}P(x)gE^{-1}g^T\left(R - \Phi(x,\xi)\right) + \frac{1}{2}F(x)^T\left(R - \Phi(x,\xi)\right) \\
M^I_{2,1} &= \left(M^I_{1,2}\right)^T \\
M^I_{2,2} &= -\frac{1}{2}\left(R - \Phi(x,\xi)\right)^T gE^{-1}g^T\left(R - \Phi(x,\xi)\right) + \\
&\quad + \left(S - \Lambda(x,\xi)\right)^T \nu \left(S - \Lambda(x,\xi)\right) + \frac{1}{2}dS \\
D_{1,1} &= +\Psi(x,\xi)\Xi(x,\xi)^T \\
D_{1,2} &= -\frac{1}{2}\Psi(x,\xi)S - \frac{1}{2}\Xi(x,\xi)R \\
D_{2,1} &= \left(D^{II}_{1,2}\right)^T \\
N^I_{1,1} &= +\Psi(x,\xi)\Psi(x,\xi)^T \\
N^I_{1,2} &= -\Psi(x,\xi)R \\
N^I_{2,1} &= \left(N^I_{1,2}\right)^T \\
N^I_{2,2} &= RR
\end{aligned}
$$

Therefore, we obtain

$$\begin{pmatrix} x \\ x-\xi \end{pmatrix}^T \left[\begin{pmatrix} -\Sigma_1(x) & M_{1,2}^I \\ M_{2,1}^I & M_{2,2}^I \end{pmatrix} - k \begin{pmatrix} N_{1,1}^I & N_{1,2}^I \\ N_{2,1}^I & N_{2,2}^I \end{pmatrix} + k \begin{pmatrix} D_{1,1} & D_{1,2} \\ D_{2,1} & D_{2,2} \end{pmatrix} \right] \begin{pmatrix} x \\ x-\xi \end{pmatrix} +$$
$$-tr\left(\left(P(x) - \bar{\Phi}(x,\xi)^T + R\right)\nu\right) - \tfrac{1}{2} dx^T \bar{\Lambda}(x,\xi)^T x - d\ln(\kappa) + \lambda \leq 0 \tag{4.62}$$

Evaluating (4.62) at $(x,\xi) = (0,0)$, noting that $\bar{\Phi}(0,0)^T = \bar{\Lambda}(0,0)^T = 0$ and substituting the proposed expression of λ i.e. (4.55) into the above expression, we have that (4.62) becomes

$$-\max_{\Omega_2}\left(-tr\left(P(x)\nu\right) - d\ln(\kappa)\right) - tr\left(P(x)\nu\right) - d\ln(\kappa) - tr\left(R\nu\right) \leq 0$$

The above expression is always verified because R and ν are positive definite and hence, by property (A.6), $-tr\left(R\nu\right)$ is strictly smaller than zero. Noting that all mappings in (4.62) are smooth we deduce that there always exists a neighborhood of the origin such that (4.62) holds and it is $\Omega_1 \cap \Omega_{alg,ext}$. Finally, the neighborhood where both the first and the second inequality of (4.50) are verified is

$$\bar{\Omega} = \Omega_1 \cap \Omega_2 \cap \Omega_{alg,ext}$$

□

Remark 4.20. Even if an explicit dependence of $u(x,\xi)$ on $m(x,\xi)$ does not appear in (4.52), the two function are strictly related. Indeed remember that $P(x)$ was computed from (4.51) where it was coupled with $G(x)$.

Remark 4.21. As Assumption 1 and 2 have to hold in a neighborhood of the origin, it is easy to see that $tr\left(\nu m_{xx}\right) \leq 0$ is equal to

$$tr\left(\nu\left(-m(x,\xi)\left((G(\xi)+S) + \left(x^T G(\xi) + (x-\xi)^T S\right)\left(x^T G(\xi) + (x-\xi)^T S\right)^T\right)\right)\right) \leq 0$$

This expression, recalling that $m(x,\xi) \geq 0$, can be rewritten as

$$-tr\left(\nu\left((G(\xi)+S) + \left(x^T G(\xi) + (x-\xi)^T S\right)\left(x^T G(\xi) + (x-\xi)^T S\right)^T\right)\right) \leq 0$$

Therefore it is always possible to choose an S such that $tr\left(\nu m_{xx}\right) \leq 0$ and, in general, Assumption 1 and Assumption 2 hold. However, when the real (i.e. not approximate) solution $m_{real}(x,\xi)$ does not satisfy Assumption 1 and Assumption 2, then our approximate solution either can not be found because for example the the algebraic solution condition does not hold or is not good because for instance $\int_{\bar{\Omega}} m_{app}(x,\xi) \ll 1$.

Remark 4.22. Note that in this proof the condition (4.51) of the algebraic MFGS is used only because we imposed $M^{I}_{1,1}$ and $M^{II}_{1,1}$ to be zero at the origin in (4.62) and (4.60). However, considering the results in [15], the structure of the algebraic MFGS could be useful to prove that the local dynamic approximate solution makes the dynamically extended mean field game (4.40) stable in a neighborhood of the origin. Anyway this aspect is not dealt with in this thesis and it may be the topic of further future works.

4.5 Numerical Example

4.5.1 Problem Description

In order to show the developed procedure we consider a simple numerical example. Imagine a school of fish. Each fish does not care about each of the other fish. Rather, it cares about how the fish nearby, as a mass, globally move. Firstly assume that fish can move wherever they want on a line and the absolute position of each fish on the line at instant t is indicated by the state $x(t) : [0,\infty[\longmapsto \mathbb{R}$. Moreover each fish controls his velocity and it depends on its own position and the position of the mass of the other fish. In other words the control function is $v(x,m,t) : \mathbb{R} \times [0,\infty[\times [0,\infty[\longmapsto \mathbb{R}$. We model the school as an infinite number of fish such that $m(x,t)$ is the probability density function that describes each fish position at instant t and $m_0(x)$ is the initial given distribution. We also assume that all fish are similar or, using the terminology of the mean field game theory, that each player has the same dynamics. Moreover they move in order to avoid being in dangerous positions or, in other words, minimizing the dangerousness of their position. For this reasons we can apply the mean field game theory in order to model this problem.

Based on what we have said, the dynamics of each fish is given by the following equation

$$\dot{x}(t) = u(x,t) + \frac{1}{10}\dot{W}(t) \quad x(0) = x_0 \tag{4.63}$$

where, according to the notation used in (4.12), we have

- the state dimension $d = 1$
- $f(x) \equiv 0$ and consequently $F(x) \equiv 0$. In a more general case $f(x)$ could be used to model the contribution of the sea current that moves the fish even if his relative velocity, i.e. the control input, is absent
- $g(x) \equiv 1$ because we are assuming that each fish can control directly its relative velocity
- $u(x,t) = v(x,m)$ in the control input that, as we said, is the relative velocity that each fish can independently choose
- x_0 is the initial position of the considered fish and it is a realization of the random variable that has $m_0(x)$ as probability density function
- $W(t)$ is a Brownian motion and it is used to model the crowding. Roughly speaking this terms models the fact that they are not in total control of their trajectories because of the presence of the other fish
- $\sigma = \frac{1}{10}$ that quantifies the contribution of the Brownian motion and that is rather large because we are considering a large school of fish where clashes may be very frequent

The cost function, that in this case is a danger index to minimized, is defined as follows

$$J\left(x_0, v(x)\right) := \lim_{T\to\infty} \frac{1}{T}\, E\left[\int_0^T \left(\frac{u^2}{2} - 100e^{-x^2}\ln\left(m\right)\right) dt\right] \tag{4.64}$$

where, according to the notation used in (4.14), we have

- $Q(x) \equiv 0$
- $E = 1$ because the term $u^2(x,m,t)$ models the fact that if a fish is moving at high speed then he risks to run out of energy

- $d(x) = 100e^{-x^2}$ because the term $-100e^{-x^2}\ln(m(x(t)))$ is used to promote an high value of $m(x,t)$ or, in other words, a situation where it is very probable that another fish has the same state value of the considered fish. However, since we want to add some nonlinearities to the cost function, we have used the function $100e^{-x^2}$, that has a maximum in the origin, in order to penalize a small value of $m(\bar{x},t)$ more if $|\bar{x}|$ is large. It models the fact that the origin is considered a safer position hence if a fish is close to the origin it is less important to be together with the others

Note that, although we consider a scalar problem in this example, the technique proposed in this work allows to deal also with d dimensional state space.

To sum up, if we assume the problem stationary, the motion of such school of fish can be modeled by the following mean field game

$$\begin{cases} \dot{x}(t) = u(x) + \frac{1}{10}\dot{W}(t) \quad x(0) = x_0 \\ J(x_0, u(x)) := \lim_{T\to\infty} \frac{1}{T} E\left[\int_0^T \left(\frac{u^2}{2} - 100e^{-x^2}\ln(m)\right) dt\right] \\ u(x) = \arg\min_{u(x)\in U_x} J(x_0, u(x)) \\ \int_{\mathbb{R}^d} m(x)dx = 1 \end{cases} \tag{4.65}$$

Note also that, although the dynamics is linear, the problem is not trivial because the cost function has some not quadratic terms that make the optimal control nonlinear on x.

4.5.2 Proposed Method Application

We want to use the proposed method to find a **local dynamical approximate** solution of (4.65).

According to the procedure proposed in Section 4.4, we first have to compute a solution $\left(\tau(x,\xi), m(x,\xi), v(x,\xi), \lambda, \bar{\Omega}\right)$ of (4.50). The latter, in this numerical example, reads

$$\begin{cases} \frac{1}{100}v_{xx} + v_\xi\dot{\xi} - \frac{1}{2}v_x^2 + \lambda - 100e^{-x^2}\ln(m) \leq 0 \\ \frac{1}{100}m_{xx} + \frac{\partial\left(m\dot{\xi}\right)}{\partial\xi} - \frac{\partial(mv_x)}{\partial x} \geq 0 \end{cases} \tag{4.66}$$

In order to solve (4.66) we want to use the results of Theorem 4.19. For this reason we need to compute an algebraic MFG solution of (4.65). The condition (4.51) reads

$$\begin{cases} -x^2P(x)G(x) - \frac{1}{100}x^2G^2(x) + P(x) + \frac{1}{100}G(x) = x^2\Sigma_2(x) \\ -\frac{1}{2}P^2(x) + 50e^{-x^2}G(x) = -\Sigma_1(x) \end{cases} \tag{4.67}$$

Note that (4.67) has to hold on a neighborhood of the origin and hence in particular at $x = 0$. Evaluating (4.67) at $x = 0$ we obtain the conditions

$$\begin{cases} P(0) = -\frac{1}{100}G(0) \\ 100G(0) \leq P^2(0) \end{cases}$$

Moreover (4.67) can be rewritten as follows

$$\begin{cases} \left(P(x) + \frac{1}{100}G(x)\right)\left(-x^2G(x) + 1\right)\frac{1}{x^2} > 0 \\ G(x) < \frac{P^2(x)}{100}e^{x^2} \end{cases}$$

It can be verified that the pair $(P(x), G(x))$

$$P(x) = x^2 \quad G(x) = \frac{1}{101}x^4e^{x^2}$$

satisfies all conditions of Definition 4.16 and hence it is an algebraic MFGS of (4.65).

Moreover, seeing Figure 4.1 where function $\Sigma_2(x)$ and $\Sigma_1(x)$ are shown, we have that they are grater than zero in the neighborhood of the origin $\Omega \simeq [-2, 2]$. Indeed conditions (4.67) holds in such neighborhood and hence $\Omega_{alg} \simeq [-2, 2]$.

According to Theorem 4.19, a solution $\left(\tau(x,\xi), m(x,\xi), v(x,\xi), \lambda, \bar{\Omega}\right)$ of (4.50) is given by (4.52), (4.53), (4.53), (4.53) and (4.53) where the values of R, S and k have to be chosen in order to make Ω_1 and Ω_2 as large as possible. We propose the following values

$$R = 15 \quad S = 2 \quad k = 10$$

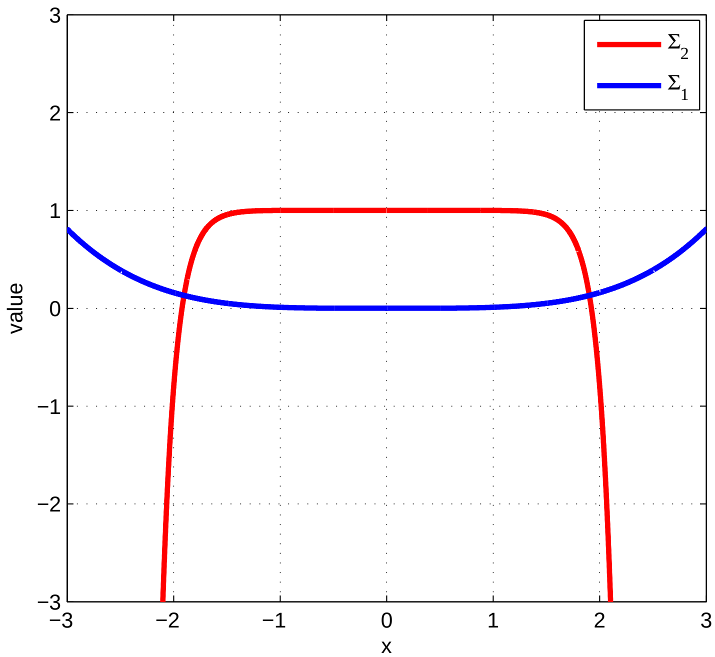


Figura 4.1: Figure of $\Sigma_1(x)$ and $\Sigma_2(x)$

Consequently, the solution $\left(\tau(x,\xi), m(x,\xi), v(x,\xi), \lambda, \bar{\Omega}\right)$ of (4.50) reads

$$v(x,\xi) = \frac{1}{2}x^2\xi^2 + \frac{15}{2}(x-\xi)^2$$

$$m(x,\xi) = 0.73\exp\left(-\frac{1}{202}x^2\xi^4e^{\xi^2} - (x-\xi)^2\right)$$

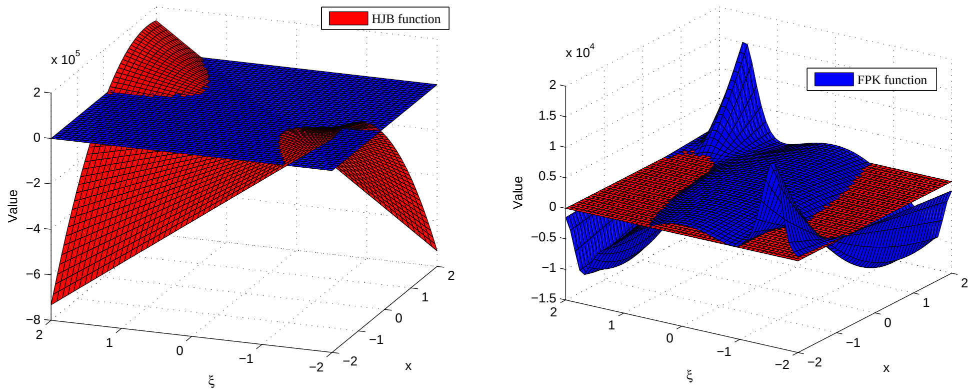
$$\dot{\xi} = \tau(x,\xi) = -10\left(x^2\xi - \frac{11}{2}(x-\xi) - \frac{1}{101}x^2\xi^3e^{\xi^2}\left(2+\xi^2\right)\right)$$

$$\lambda \;\; = - \max_{\Omega_2\cap\Omega_{alg,ext}}\left(-\frac{x^2}{100} - 100e^{-x^2}\ln(0.73)\right)$$

$$\bar{\Omega} = \Omega_1 \cap \Omega_2 \cap \Omega_{alg,ext}$$

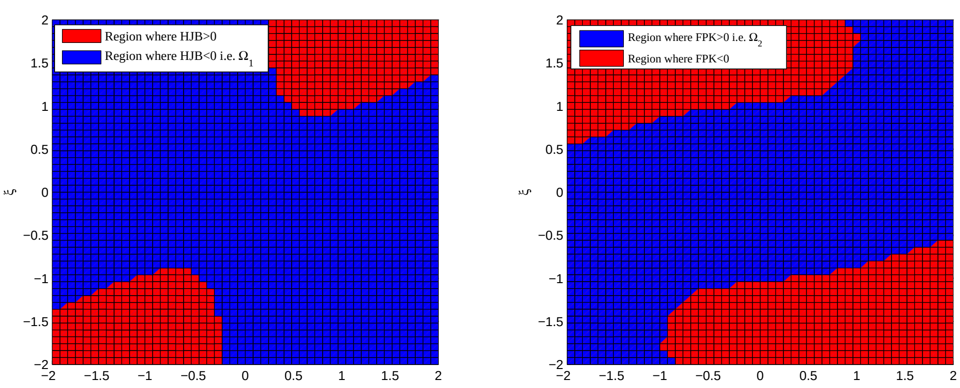
where the corresponding HJB function and FPK function are shown in Figure 4.2a and 4.2b respectively. Consequently the neighborhoods of the origin

$\Omega_1 \cap \Omega_{alg,ext}$ and $\Omega_2 \cap \Omega_{alg,ext}$ where such inequalities hold, w.r.t. $-2 < \xi < 2$, are available in Figure 4.3a and 4.3b respectively.



(a) HJB function (b) FPK function

Figure 4.2: HJB and FPK inequality



(a) Neighborhood of the origin Ω_1 (b) Neighborhood of the origin Ω_2

Figure 4.3: Ω_1 and Ω_2

According to (4.42), the local dynamic approximate solution of (4.65) is therefore given by $(\hat{\tau}(x,\xi), \hat{m}(x,\xi), \hat{u}(x,\xi))$ where

$$\hat{\tau}(x,\xi) = \tau(x,\xi) \quad \hat{m}(x,\xi) = m(x,\xi) \quad \hat{u}(x,\xi) = -\left(x\xi^2 + 15\,(x-\xi)\right)$$

Moreover the value function $v(x,\xi)$ and the population density function $m(x,\xi)$ are shown in Figure 4.4 and 4.5.

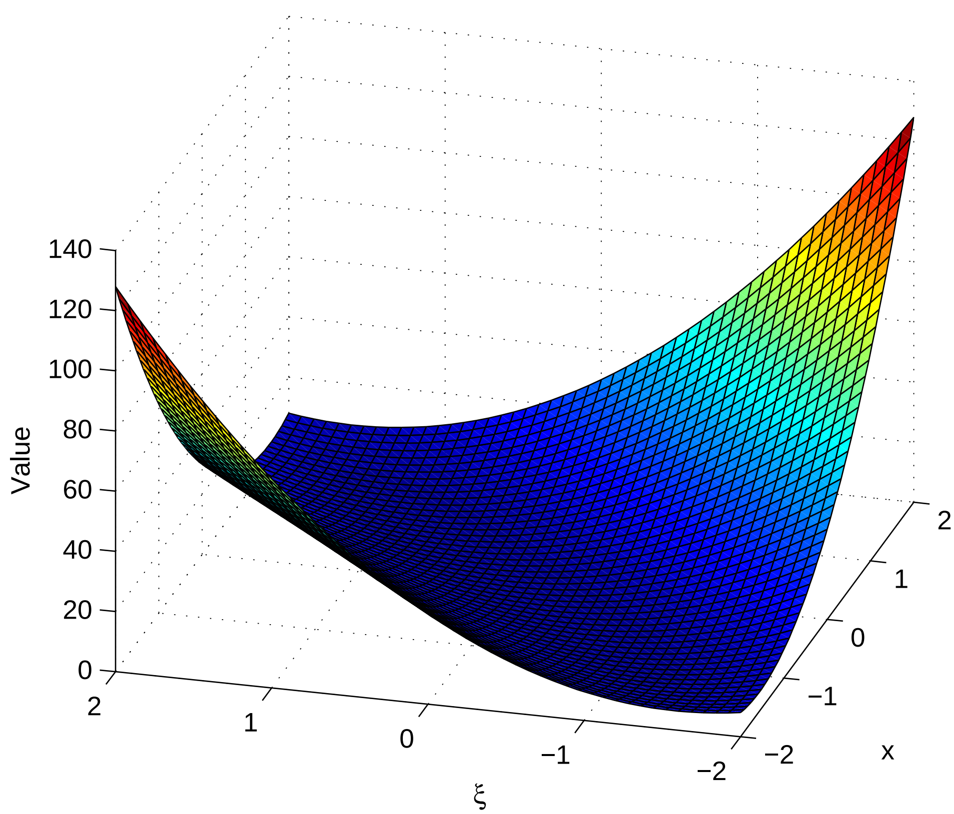


Figura 4.4: Value function $v(x,\xi)$

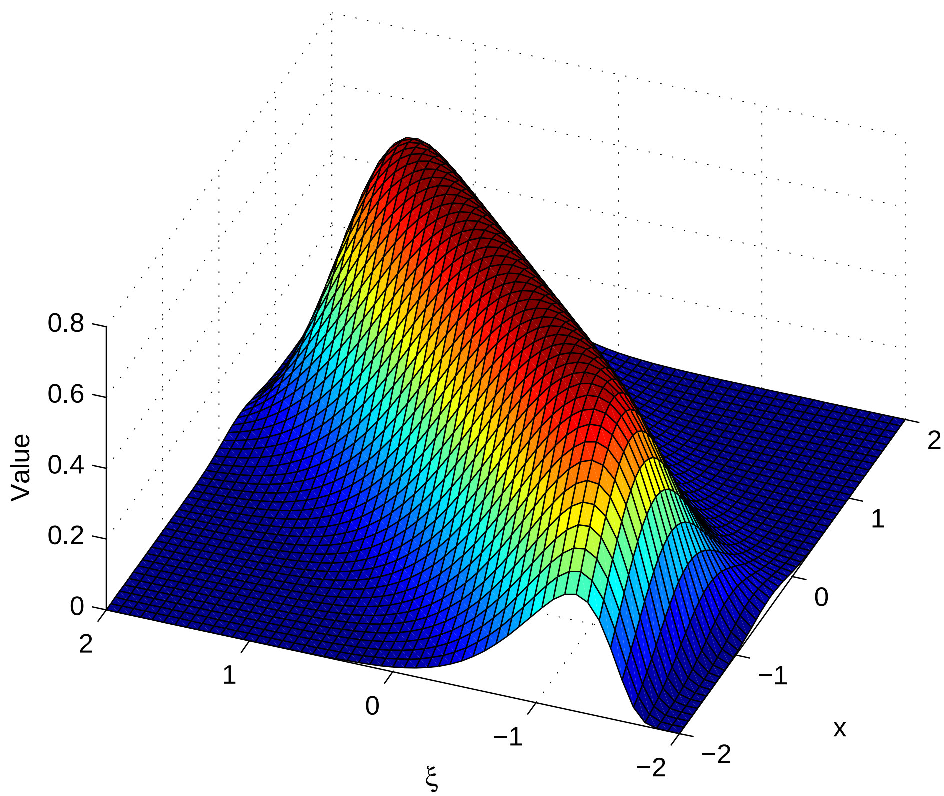
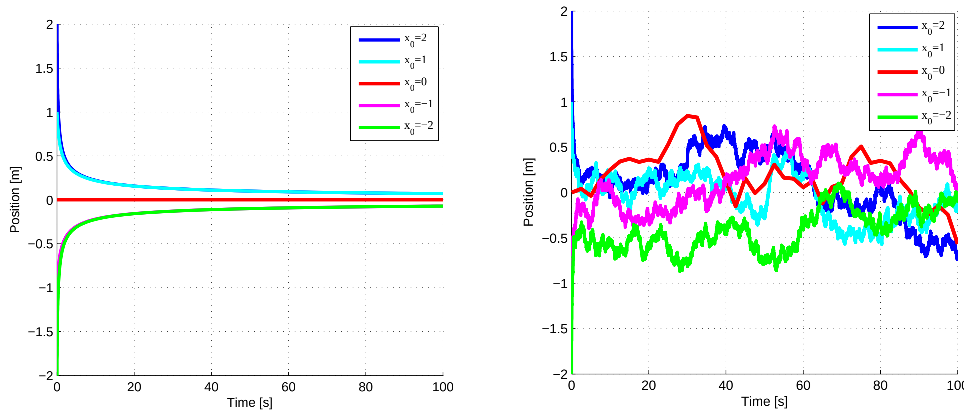


Figura 4.5: Population density function $m(x, \xi)$

In order to show the performances of the proposed solution, according to the procedure described in Section 4.4, we consider the following dynamically extended mean field game

$$\begin{cases} \dot{x}(t) = \hat{u}(x,t) + \frac{1}{10}\dot{W}(t) \quad x(0) = x_0 \\ \dot{\xi}(t) = \hat{\tau}(x,\xi) \quad \xi(0) = \xi_0 \\ J\left(x_0, \xi_0, \hat{u}(x)\right) := \lim_{T\to\infty} \frac{1}{T}\, E\left[\int_0^T \left(\frac{\hat{u}^2}{2} - 100e^{-x^2} \ln\left(m\right)\right) dt\right] \end{cases}$$

We analyze first the state trajectory $x(t)$. We are interesting only in this first part of the extended state $(x(t), \xi(t))$ because it has a physical meaning. In particular, in the model that we have proposed, it is the position of a



(a) Simulation of the behavior of $x(t)$ **without** the contribution of the Brownian motion $\dot{W}(t)$

(b) Simulation of the behavior of $x(t)$ **with** the contribution of the Brownian motion $\dot{W}(t)$

Figure 4.6: Behavior of $x(t)$ with and without the contribution of the Brownian motion and with different initial conditions x_0

particular fish on the line. In Figure 4.6a the behavior of the state $x(t)$ with different initial conditions x_0 and $\xi_0 = 0$ without the contribution of the Brownian motion (i.e. $\sigma = 0$) is shown. All the state trajectories converge on the origin. This is due to the fact that the value function in Figure 4.4 is concave everywhere and has a minimum at the origin. For this reason, the corresponding feedback control $\hat{u}(x, \xi)$ is such that each player is attracted towards the origin. We also note that this behavior of the trajectory of each fish, when they are moving according to the suboptimal solution, is consistent. Indeed, remembering the form of the cost function (4.64), we have that the value of the state that minimizes the cost is $(x, \xi) = \left(0, \bar{\xi}\right)$ for each $\bar{\xi} \in \mathbb{R}$. Moreover in Figure 4.6a we also note that the derivative of $x(t)$, i.e. the velocity, has a very large norm when the corresponding player is far from the origin. However the velocity norm is smaller when the player is close to the origin. This is consistent indeed in the cost function (4.64) there is a term, i.e. $\frac{u^2}{2}$, that penalizes an large velocity and another term, i.e. $-100e^{-x^2} \ln(m)$, that penalizes the fact that a fish is far from the other fish or far from the origin. When a fish is far from the origin the latter term is the biggest and, because of this, it moves towards origin very fast. On the contrary, when a fish is near to the origin, the first term of the cost function is the biggest and, because of this, the velocity of the fish decreases. Analyzing these first results we could say that the local dynamic approximate solution of (4.65) seems to be accurate.

In Figure 4.6b some realizations of $x(t)$ with different initial conditions x_0 and $\xi_0 = 0$ are available. We have to consider realizations of $x(t)$ because $x(t)$ is influenced by the Brownian motion that is a stochastic process and therefore it is a stochastic process. Differently from Figure 4.6a , in Figure 4.6b we observe that the norm of $x(t)$ does not decrease with the increase of time t. It decreases only in the initial instants or, in other words, when it describes a fish that is far from the origin. This is due to the Brownian motion that plays the role of a noise. It is countered by feedback control only when the norm of x is large and therefore the cost function (4.64) is large too. On the contrary when the norm of x is smaller the contribution of the feedback control in the cost function is smaller and for this reason the contribution of the Brownian motion is sufficient to get the considered player away from the origin even if the origin is the position that minimizes the cost function. However, this behavior of the state trajectories $x(t)$ and consequently of the fish that we are modeling is consistent. Indeed, in a real framework, it is impossible that all fish go simultaneously to the same point firstly because they can not physically occupy the same exact position and secondly because each fish can not control his own position exactly because of the presence of other fish and the possible collisions. In other words, as we said, Brownian motion in this case models the crowding. This is an example of the fact that the Brownian motion is often very useful to make a mean field game suitable to model real phenomena. Finally we note that, at least in the realizations that we are considering in Figure 4.6b, for a large t fish stay in a neighborhood of the origin such that $-1 < x < 1$. However this behavior is different from the behavior described by the population density function $m(x, \xi)$ in Figure 4.5. Indeed in Figure 4.5 we see that each position such that $-2 < x < -2$ has approximately the same probability to be occupied by a fish. This mismatch may be due either to the fact that we have considered realizations that hence that do not describe the general behavior of players or to the fact that the proposed solution is only an approximation.

Finally, we can say that, in this simple numerical example the local dynamic approximate solution describes a behavior of players that is very similar to what we expect.

Chapter 5

Conclusions and Future Works

In this work the theory of mean field games has been introduced. The meaning of the HJB PDE and of the FPK PDE, that have to be solved in order to find the optimal solution of a mean field game problem, have been studied. Moreover, a method to find a local approximate dynamic solution for a class of stationary mean field games has been proposed. It is based on the approach used in other methods developed for optimal control and differential games. However, new ideas are considered in order to deal with the structure of the involved equation. This procedure has been finally applied to an easy numerical problem and the produced results have proven to be consistent.

The positive aspects of the proposed procedure to solve mean fields games are the following

- It allows to solve a rather large class of mean field games simply solving PDIs and avoiding dealing with PDEs
- It allows to always find a suboptimal solution of a class of stationary mean field games
- It provides consistent results at least for simple problems

Nevertheless, this work is thought of being a first step and it can be extended for example studying a more general solution of a larger class of nonlinear mean field games. Indeed, some of the limits oh the proposed strategy are the following

- It is local and hence it provides a suboptimal solution that holds only in a generally small neighborhood of the origin. This limit could be removed by deeply studying the structure of the inequalities (4.60) and (4.62) and trying to find some assumptions that allow to solve them for each $(x, \xi) \in \mathbb{R}^{2d}$

- It is an approximate solution and hence the corresponding optimal control does not allow to minimize the cost function. This problem can not be solved because the only way to find an optimal solution for a mean field game in the general case is to solve the HJB and FPK PDEs. However it is typically very difficult. Nevertheless, it could be possible to find some conditions that allow to quantified the accuracy of the proposed solution and to increase it. For example in [15] the the accuracy of the local dynamic approximate solution can be increased acting on the initial condition of the extended state ξ_0 whose role has not been studied in this work

- It works only on a small class of nonlinear mean field games. It could be made larger by considering, fore instance, a more complex form of the cost function (4.14) with some terms linked to the expectation and the variance of the population density function $m(x)$.

- It work only on stationary mean field games but it could be extended to non stationary mean field games by introducing the dependence on time and therefore considering the general HJB FPK PDEs (4.5).

- The local dynamic approximate solution could make the mean field games unstable. It may be avoided by deeply studying the behavior of the mean field game in a neighborhood of the origin and in particular, considering its linearization around the origin and providing results in [4].

Appendix A

Mathematical tools

In this appendix we introduce some mathematical tools that are widely used in the text.

Vector derivatives

Let $c \in \mathbb{R}^n$ be a vector, $A \in \mathbb{R}^{n \times n}$ a symmetric matrix, $x = \begin{pmatrix} x_1 & \dots & x_n \end{pmatrix}^T$ and $W(x) : \mathbb{R}^n \longmapsto \mathbb{R}^{n \times n}$ a smooth function. Then the following properties hold

1.
$$\frac{\partial C^T x}{\partial x} = c^T \tag{A.1}$$

2.
$$\frac{\partial Ax}{\partial x} = A \tag{A.2}$$

3.
$$\frac{\partial x^T Ax}{\partial x} = x^T \left(A + A^T \right) = 2x^T A \tag{A.3}$$

4.
$$tr\left(\frac{\partial x^T W(x)}{\partial x} \right) = tr\left(W(x) \right) + x^T \bar{\nabla}_x \left(W(x) \right) \tag{A.4}$$

Proof.

$$
\begin{aligned}
tr\left(\frac{\partial x^T W(x)}{\partial x}\right) &= tr\left(\frac{\partial \begin{pmatrix} \sum_{i=1}^n x_i W_{i,1}(x) & \cdots & \sum_{i=1}^n x_i W_{i,n}(x) \end{pmatrix}}{\partial x}\right) \\
&= tr\begin{pmatrix} \sum_{i=1}^n \left(x_i \frac{\partial(W_{i,1})}{\partial x_1}\right) + W_{1,1} & \cdots & \sum_{i=1}^n \left(x_i \frac{\partial(W_{i,n})}{\partial x_1}\right) + W_{1,n} \\ \vdots & \ddots & \vdots \\ \sum_{i=1}^n \left(x_i \frac{\partial(W_{i,1})}{\partial x_n}\right) + W_{n,1} & \cdots & \sum_{i=1}^n \left(x_i \frac{\partial(W_{i,n})}{\partial x_n}\right) + W_{n,n} \end{pmatrix} \\
&= tr\left(W\right) + tr\begin{pmatrix} \sum_{i=1}^n \left(x_i \frac{\partial(W_{i,1})}{\partial x_1}\right) & \cdots & \sum_{i=1}^n \left(x_i \frac{\partial(W_{i,n})}{\partial x_1}\right) \\ \vdots & \ddots & \vdots \\ \sum_{i=1}^n \left(x_i \frac{\partial(W_{i,1})}{\partial x_n}\right) & \cdots & \sum_{i=1}^n \left(x_i \frac{\partial(W_{i,n})}{\partial x_n}\right) \end{pmatrix} \\
&= tr\left(W(x)\right) + x^T \bar{\nabla}_x \left(W(x)\right)
\end{aligned}
$$

□

Trace properties

Let $B, C \in \mathbb{R}^{n\times n}$ be matrices, $P \in \mathbb{R}^{n\times n}$ a positive definite matrix and $N \in \mathbb{R}^{n\times n}$ a negative definite matrix. Then the following properties hold

1. $$tr(BC) = tr(CB) \tag{A.5}$$

2. $$tr(P) > 0 \qquad tr(N) < 0 \tag{A.6}$$

3. $$tr(PP) > 0 \qquad tr(PN) < 0 \tag{A.7}$$

Proof. We prove that $tr(PN) < 0$. Exploiting the Choleshy decomposition we have that

$$P = P^{\frac{1}{2}} P^{\frac{T}{2}}$$

Exploiting property (A.5) then

$$tr\,(PN) = tr\,(NP) = tr\left(NP^{\frac{1}{2}}P^{\frac{T}{2}}\right) = tr\left(P^{\frac{T}{2}}NP^{\frac{1}{2}}\right)$$

Considering the vector $z = P^{\frac{1}{2}}x$ we have that $x^T P^{\frac{T}{2}} N P^{\frac{1}{2}} x = z^T N z < 0$ for each $z \in \mathbb{R}^n$ and hence for each $x \in \mathbb{R}^n$. For this reason the matrix $P^{\frac{T}{2}} N P^{\frac{1}{2}}$ is negative definite. The proof is completed using property (A.6). □

Divergence properties

Given $F(x) : \mathbb{R}^n \longmapsto \mathbb{R}^{1\times n}$, $\varphi(x) : \mathbb{R}^n \longmapsto \mathbb{R}$, $a \in \mathbb{R}$, $A \in \mathbb{R}^{n\times n}$, some properties of the divergence operator are

1.
$$div\,(\varphi(x)F(x)) = F(x)\varphi_x^T(x) + \varphi(x)\; div(F(x)) \tag{A.8}$$

2.
$$div\,(aF(x)) = a\; div(F(x)) \tag{A.9}$$

3.
$$div\,(\varphi_x(x)A) = tr\,(\varphi_{xx}(x)A) = tr\,(A\varphi_{xx}(x)) \tag{A.10}$$

Appendix B

Proofs

B.1 Theorem 2.10

Proof. Equation (2.12) is equivalent to the following system of inequalities that holds for any $\varepsilon > 0$ and $h \geq 0$

$$\begin{cases} V(x_0, t_0) \leq \inf_{u(t)\in U} \left(\int_t^{t+h} L\left(x(s), u(t), t\right) ds + V(x(t_0 + h), t + h) \right) + \varepsilon \\ V(x_0, t_0) \geq \inf_{u(t)\in U} \left(\int_t^{t+h} L\left(x(s), u(t), t\right) ds + V(x(t_0 + h), t + h) \right) - \varepsilon \end{cases} \tag{B.1}$$

where $x(s) = x(s; t_0, x_0, u(t))$ and $x(t_0 + h) = x(t_0 + h; t_0, x_0, u(t))$. The first inequality is an upper bound and the second one is a lower bound for $V(x_0, t_0)$. We consider first the upper bound and a particular $V(\hat{x}, \hat{t})$ where $\hat{x} \in \mathbb{R}^n$ and $\hat{t} \geq 0$.

Choose any $u_1(t) \in U$, where U is the set of the assumption **(H1),** and define the corresponding trajectory $x_1(t)$ according to (2.3). We have that

$$\dot{x}_1(t) = f\left(x_1(t), u_1(t)\right) \qquad x_1(\hat{t}) = \hat{x}$$

for $t > \hat{t}$. Moreover $x_1(t)$ exists and is unique because **(H1)** holds. Fix $\varepsilon > 0$ and choose $u_2(t) \in U$ such that

$$V(x_1(\hat{t} + h), \hat{t} + h) + \varepsilon \geq \lim_{T\longrightarrow\infty} \int_{\hat{t}+h}^{T} L\left(x_2(t), u_2(t), t\right) dt$$

where

$$\dot{x}_2(t) = f\left(x_2(t), u_2(t)\right) \qquad x_2(\hat{t} + h) = x_1(\hat{t} + h)$$

for $t > \hat{t} + h$. Note that such a $u_2(t)$ exists because **(H2)** holds. Define now a new control $u_3(t)$

$$u_3(t) = \begin{cases} u_1(t) & \text{if } t \in [\hat{t}, \hat{t}+h[\\ u_2(t) & \text{if } t \in [\hat{t}+h, \infty[\end{cases}$$

which gives rise to trajectory

$$\dot{x}_3(t) = f\left(x_3(t), u_3(t)\right) \qquad x_3(\hat{t}) = \hat{x}$$

for $t > \hat{t}$. Once again the solution of (2.3) is unique because **(H1)** holds

$$x_3(t) = \begin{cases} x_1(t) & \text{if } t \in [\hat{t}, \hat{t}+h[\\ x_2(t) & \text{if } t \in [\hat{t}+h, \infty[\end{cases}$$

Consequently

$$\begin{aligned} V(\hat{x}, \hat{t}) &\leq J\left[u_3(t), \hat{x}, \hat{t}\right] \\ &= \lim_{T \longrightarrow \infty} \int_{\hat{t}}^{T} L\left(x_3(t), u_3(t), t\right) dt \\ &= \int_{\hat{t}}^{\hat{t}+h} L\left(x_1(t), u_1(t), t\right) dt + \lim_{T \longrightarrow \infty} \int_{\hat{t}+h}^{T} L\left(x_2(t), u_2(t), t\right) dt \\ &\leq \int_{\hat{t}}^{\hat{t}+h} L\left(x_1(t), u_1(t), t\right) dt + V(x_1(\hat{t}+h), \hat{t}+h) + \varepsilon \end{aligned}$$

Since $u_1(t)$ is arbitrary, it must hold that

$$V(\hat{x}, \hat{t}) \leq \inf_{u(t) \in U} \left[\int_{\hat{t}}^{\hat{t}+h} L\left(x(t), u(t), t\right) dt + V(x(\hat{t}+h), \hat{t}+h) \right] + \varepsilon$$

Considering now the lower bound, fix $\varepsilon > 0$ and choose $u_4(t) \in U$ such that

$$V(\hat{x}, \hat{t}) \geq \lim_{T \longrightarrow \infty} \int_{\hat{t}}^{T} L\left(x_4(t), u_4(t), t\right) dt - \varepsilon$$

where

$$\dot{x}_4(t) = f\left(x_4(t), u_4(t)\right) \qquad x_4(\hat{t}) = \hat{x}$$

for $t > \hat{t}$. From the definition of value function

$$V(x_4(\hat{t}+h), \hat{t}+h) \leq \lim_{T \longrightarrow \infty} \int_{\hat{t}+h}^{T} L\left(x_4(t), u_4(t), t\right) dt$$

Finally

$$V(\hat{x}, \hat{t}) \geq \inf_{u(t) \in U} \left[\int_{\hat{t}}^{\hat{t}+h} L\left(x(t), u(t), t\right) dt + V(x(\hat{t}+h), \hat{t}+h) \right] - \varepsilon$$

□

B.2 Theorem 2.11

Proof. Since **(H1)** and **(H2)** hold, the dynamic programming principle holds. Thus we can rearrange (2.12) as follows

$$\inf_{u(t) \in U} \left[\int_{t}^{t+h} L\left(x(s), u(s), s\right) ds + V(x(t+h), t+h) - V(x,t) \right] = 0$$

where $h > 0$, $t > 0$ and $u(t) \in U$ is an input and $x(t)$ is the corresponding unique solution of (2.3). Divide through by $h > 0$

$$\inf_{u(t) \in U} \left[\frac{1}{h} \int_{t}^{t+h} L\left(x(s), u(s), s\right) ds + \frac{V(x(t+h), t+h) - V(x,t)}{h} \right] = 0$$

Let $h \longrightarrow 0$ and, on the region Ω where V is differentiabled, we have

$$\inf_{u(t) \in U} \left[L\left(x(t), u(t), t\right) + \frac{d}{dt} V(x,t) \right] = 0$$

Applying the chain rule on $\frac{d}{dt} V(x,t)$

$$\inf_{u(t) \in U} \left[L\left(x(t), u(t), t\right) + V_t(x,t) + V_x(x,t) \frac{d}{dt} x(t) \right] = 0$$

Then, we can substitute the system dynamics $\frac{d}{dt}x(t) = \dot{x}(t) = f(x, u, t)$

$$\inf_{u(t)\in U} \left[L\left(x(t), u(t), t\right) + V_t(x, t) + V_x(x, t) f(x(t), u(t), t)\right] = 0$$

Observe that the only dependence on $u(t) \in U$ is $u(t) = u \in \mathbb{R}^m$

$$\inf_{u\in \mathbb{R}^m} \left[L\left(x, u, t\right) + V_t(x, t) + V_x(x, t) f(x, u, t)\right] = 0$$

Thus we obtain the HJB PDE

$$V_t(x, t) + H(x, u, t, V_x(x, t)) = 0$$

where $H(x, u, t, V_x(x, t))$ is the Hamiltonian with $p(t) = V_x(x, t)$. This substitution can be made because, according to the notation adopted, both $p(t)$ and $V_x(x, t)$ are row vectors with the same dimension. $\square$

Bibliography

[1] J. M. Lasry, P.L. Lions, *Mean Field games*, Japanese Journal of Mathematics, 2, March 2007

[2] M. Huang, P.E. Caines, R.P. Malhame, *Large-population cost-coupled LQG problems with nonuniform agents: individual-mass behavior and decentralized-Nash equilibria*, IEEE Trans. Automat. Control 52 pp 1560-1571, 2007

[3] O. Gueant. *A reference case for mean field games*, Journal de Mathematiques Pures et Appliquees, 2009.

[4] M. Bardi, *Explicit solutions of some linear-quadratic mean field games*, American Institute of Mathematical Sciences, pp. 243 - 261, Volume 7, Issue 2, June 2012

[5] T.Mylvaganam, D.Bauso, A.Astolfi, *Mean field games and two–point boundary value problems,* Decision and Control (CDC), 2014 IEEE 53rd Annual Conference, pages 2722 - 2727, Dec. 2014

[6] A. Bensoussan, K.C.J.Sung, S.C.P.Yam, S.P.Yung, *Linear-Quadratic Mean Field Games*, Apr. 2014

[7] T. Basar, G.J. Olsder, *Dynamic Noncooperative Game Theory*, Classics in Applied Mathematics, 1999

[8] F. Ticozzi, *Lecture notes on Nonlinear System and Control.* 2015

[9] H. J. Pesch, M. Plail, *The Maximum Principle of optimal control: A history of ingenious ideas and missed opportunities,* Control and Cybernetics, vol. 38, No 4A, 2009

[10] V. Arnold, *Ordinary Differential Equations*. Springer-Textbook, New York, 1992

[11] J. D. Meiss, *Differential dynamical systems*, vol. 14. Siam, 2007

[12] L. Cesari, *Optimization – theory and applications*. Problems with ordinary differential equations, Springer Verlag, New York, 1983

[13] L.S. Pontryagin, V.G Boltyanskii, R.V. Gamkrelidze, E.F. Mishchenko, *The mathematical theory of optimal processes*, Interscience Publishers, John Wiley & Sons, New York, 1962.

[14] A. D. Lewis, *The Maximum Principle of Pontryagin in control and in optimal control*, Queen's University, Kingston, 2006

[15] M. Sassano, A. Astolfi, *"Dynamic approximate solutions of the HJ inequality and of the HJB equation for input-affine nonlinear systems*," IEEE Trans. Autom. Control, vol. 57, no. 10, pp. 2490–2503, Oct. 2012.

[16] T. Mylvaganam, M. Sassano, A.Astolfi *Constructive ε-Nash Equilibria for Nonzero-Sum Differential Games,* IEEE Trans. Autom. Control, vol. 60, no. 4, pp. 950–965, Apr. 2015.

[17] S. Aseev, A. Kryazhimskii, *The Pontryagin Maximum Principle for Infinite-Horizon Optimal Controls,* International Institute Applied Systems Analysis, Laxenburg, Austria, Apr. 2003.

[18] J. Yong, *Differential Games, A concise introduction,* World scientific Publishing Co., 2015

[19] A. Bressan, B. Piccoli, *Introduction to the mathematical theory of control*, vol. 2. American institute of mathematical sciences Springfield, 2007

[20] J. Sondik, *The Optimal Control of Partially Observable Markov Processes over the Infinite Horizon: Discounted Costs*, Operations Research, pp. 282-304, Apr. 1978

[21] H. Halkin, *Necessary conditions for optimal control problems with infinite horizons*, Econometrica, vol. 42, No. 2, pp. 267–272, 1974.

[22] D. A. Carlson, A. B. Haurie, A. Leizarowitz, *Infinite Horizon Optimal Control,* Springer-Verlag, Berlin Heidelberg 1991

[23] I. N. Katz, H. Mukai, H. Schattler, M. Zhang, *Solution of a differential game formulation of military air operations by the method of characteristics*, in Proc. Amer. Control Conf., 2001, vol. 1, pp. 168–175.

[24] E. Dockner, N. S. G. Jorgensen, S. Van Long, *Differential Games in Economics and Management Science*, Cambridge, U.K.: Cambridge Univ. Press, 2000

[25] J. Hofbauer, K. Sigmund, *Evolutionary game dynamics*, Bull. Amer. Math. Soc., vol. 40, pp. 479–519, 2003.

[26] T. Mylvaganam, A. Astolfi, *Approximate optimal monitoring: Preliminary results*, in Proc. Amer. Control Conf., 2012, pp. 4745–4750

[27] R. D. Lute, H. Raivfa, *Games and Decisions*, John Wiley and Sons, New York, 1957

[28] A .W. Starr, Y.C. Ho, *Nonzero-Sum Differential Games,* Journal of optimization theory and applications, vol.3, No. 3, 1969

[29] R. Isaacs, *Differential Games: A Mathematical Theory With Applications toWarfare and Pursuit, Control and Optimization.* New York, NY, USA: Dover, 1999.

[30] A. J. Krener, *The existence of optimal regulators*, in Proc. 37th Conf. Decision Control, Tampa, FL, 1998, vol. 3, pp. 3081–3086

[31] A. Wernli, G. Cook, "Successive control for the nonlinear quadratic regulator problem," Automatica, vol. 11, pp. 75–84, 1975

[32] M. G. Crandall, L. C. Evans, P.-L. Lions, *Some properties of viscosity solutions of Hamilton-Jacobi equations,* Trans. Amer. Math. Soc. 282 (1984), 487-502

[33] I. M. Mitchell, *Optimal Control & Viscosity Solutions,* Ban International Research Station Workshop 11w5086*, 2011*

[34] M. Bardi, I. Capuzzo-Dolcetta, *Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations,* Springer Science LLC, 1997

[35] D. Brian, D. O. Anderson, B. Moore, *Optimal Control: Linear Quadratic Methods, Dover Publications*, Inc., Mineola, New York, 2014

[36] M. G. Crandall, P. L. Lions, *Two Approximations of Solutions of Hamilton-Jacobi Equations,* Mathematics of Computation, vol. 43, num. 167, pg. 1-19, July 1984

[37] E. N. Barron, R. Jensen, W. Liu, Explicit solution of some first-order PDE's, Journal of Dynamical and Control Systems, Volume 3, Issue 2, pp 149-164, June 1997

[38] K. M. Anstreicher and M. H. Wright, *A note on the augmented Hessian when the reduced Hessian is semidefinite*, SIAM J. Optimiz., vol. 11, no. 1, pp. 243–253, 2000

[39] Y. Huang, W. M. Lu, *Nonlinear optimal control: Alternatives to Hamilton-Jacobi equation*, in Proc. 35th Conf. Decision Control, Kobe, Japan, Dec. 1996, vol. 4, pp. 3942–3947.

[40] A. Cournot, N. Bacon, and I. Fisher, *Researches Into the Mathematical Principles of the Theory of Wealth.* Economic classics, Macmillan, 1897

[41] J. F. Nash et al., *Equilibrium points in n-person games*, Proceedings of the national academy of sciences, vol. 36, no. 1, 1950

[42] A. Bensoussan, *Explicit solutions of linear quadratic differential games,* Stochastic Processes, Optimization and Control Theory. Applications in Financial Engineering, Queueing Networks and Manufacturing Systems, chp 2, 2006

[43] Y. Achdou, F. Camilli, I. Capuzzo-Dolcetta, *Mean field games: numerical methods for the planning problem.* 2010.

[44] F. S. Priuli, *Linear-Quadratic N-Person and Mean-Field Games: Infinite Horizon Games with Discounted Cost and Singular Limit*s, Springer Science+Business Media, New York 2014

[45] P. Cardaliaguet, M. Lasry, L. Lions, A. Porretta, *Long time average of mean field games,* Netw. Heterog. Media 7*,* pages 279-301, 2012

[46] A. Gomes, L. Nurbekyan, A. Pimentel, *Economic Models and Mean field Games Theory,* Coloquio Brasileiro de Matematica, 2015

[47] C. W. Gardiner, *Handbook of Stochastic Methods,* Springer, 1997

[48] B. Kappen, *Stochastic optimal control theory,* Springer, Aug. 2012

[49] G. Suárez, M. Hoyuelos, H. Mártin, *Mean field approach for diffusion of interacting particles,* American Physical Society, Sept. 2015

[50] D. Bauso, B. M. Dia, B. Djehiche, H. Tembine, R. Tempone, *Mean-Field Games for Marriage.* PLoS ONE 9(5), 2014

[51] D. Bauso, T. Mylvaganam, A. Astolfi, *Crowd-Averse Robust Mean-Field Games: Approximation via State Space Extension, 2013*

[52] C. Dogbe, *Modeling crowd dynamics by the mean-field limit approach,* Mathematical and Computer Modelling, Volume 52, Issues 9–10, Nov. 2010

[53] P. Ring, *Relativistic mean field theory in finite nuclei,* Prog. Part.Nucl.Phys, Vol.37, pp193-263,1996

[54] J. Engwerda, *Linear Quadratic Differential Games: An Overview,* Annals of the International Society of Dynamic Games pp 1-34, Mar. 2009